"Skein and cluster algebras of marked surfaces without punctures for $\mathfrak{sl}_3$"

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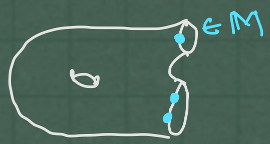
joint work with Tsukasa Ishibashi (RIMS)

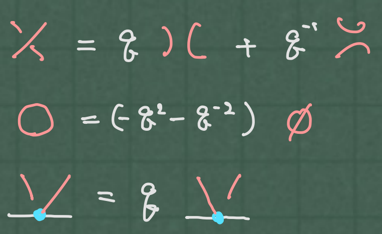
(arXiv : 2101.00643)

- $\mathfrak{g} = \mathfrak{sl}_2$ $\quad \Sigma = (\Sigma, M) =$

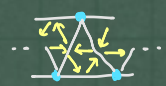


Knot theory — knots on Σ

the Kauffman bracket skein algebra $\mathscr{S}_\Sigma$
(Muller's generalization)



Representation theory — triangulations (quiver)



the cluster algebra $\mathscr{A}_{s(\mathfrak{sl}_2,\Sigma)}$

$\cap$ — Laurent expression

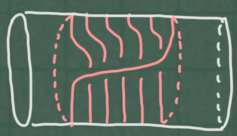
the upper cluster algebra $\mathscr{U}_{s(\mathfrak{sl}_2,\Sigma)}$

<u>Thm</u> (Muller, 2016)

$$\mathscr{A}_{s_q(\mathfrak{sl}_2,\Sigma)} \subset \mathscr{S}_\Sigma[\partial^{-1}] \subset \mathscr{U}_{s_q(\mathfrak{sl}_2,\Sigma)} \xrightarrow{\ \&\ \mathscr{A} = \mathscr{U}\ (\# M \geq 2)\ } \mathscr{A}_{s_q(\mathfrak{sl}_2,\Sigma)} = \mathscr{S}_\Sigma[\partial^{-1}] = \mathscr{U}_{s_q(\mathfrak{sl}_2,\Sigma)}$$

skein algebra		quantum cluster algebra
• web cluster	←----→	• cluster
• elementary web	←----→	• cluster variables $\{A_i\}$
• skein relation	←----→	• quantum exchange relation
• bracelet basis	←----→	• theta basis

(strong) Laurent positivity



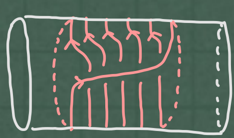
Main theorem [Ishibashi - Y]

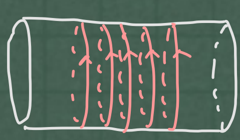
Laurent phenomenon [Berenstein - Zelevinsky '05]

- $\mathscr{S}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \subset \mathscr{A}_{\mathfrak{S}_q(\mathfrak{sl}_3,\Sigma)}$ $\left(\subset \mathcal{U}_{\mathfrak{S}_q(\mathfrak{sl}_3,\Sigma)} \right)$
- $\mathscr{S}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \xrightarrow[\text{Laurent expansion}]{} \mathcal{U}_{\mathfrak{S}_q(\mathfrak{sl}_3,\Sigma)}$

elevation preserving webs $\longmapsto$ positive Laurent polynomials on the cluster $\mathscr{C}_\Delta$

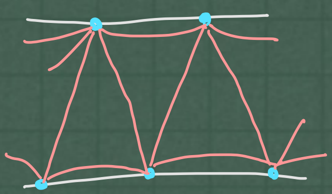
$\cup$

bracelets  $\longmapsto$ $\mathbb{Z}_+[q^{\pm 1}][e^{\pm 1} \mid e \in \Delta]$

bangles 

Ex. ($\mathfrak{sl}_2$) $\mathscr{S}_\Sigma[\partial^{-1}] = \mathscr{A}_{\mathscr{S}_{\mathfrak{g}}(\Sigma, \mathfrak{sl}_2)}$

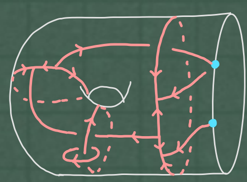
web cluster $\mathscr{C}_\Delta = \{ e \mid e : \text{edge of } \Delta \} \subset \mathscr{S}_\Sigma[\partial^{-1}]$



$ee' = A^{\circledast} e'e \quad (e, e' \in \mathscr{C}_\Delta)$

$\rightsquigarrow \mathscr{C}_\Delta$: a quantum torus

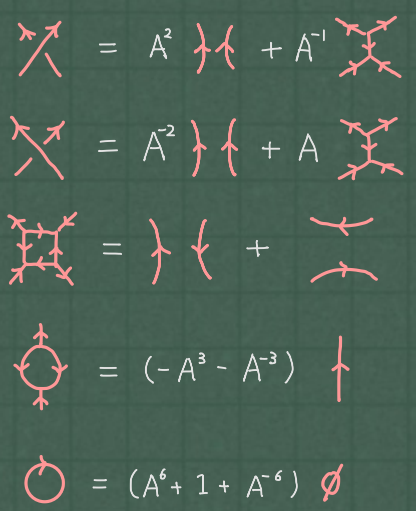
Definition (the $\mathfrak{sl}_3$ - skein algebra of Σ)



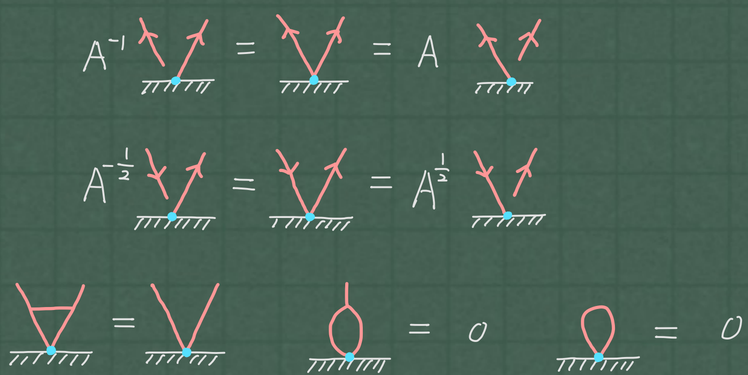
$\mathscr{S}_{\mathfrak{sl}_3, \Sigma} := \mathrm{span}_{\mathbb{Z}_A} \left\{ \begin{matrix} \text{tangled trivalent} \\ \text{graphs on } \Sigma \end{matrix} \right\} \Big/$ (1) $\mathfrak{sl}_3$ - skein relations
(2) boundary $\mathfrak{sl}_3$ - skein relations
(3) isotopy of Σ rel. to $\partial\Sigma$

$\mathbb{Z}[A^{\pm \frac{1}{2}}]$ $\quad$ $\mathfrak{sl}_3$-webs

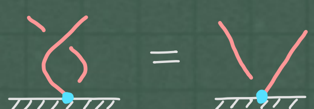
(1) $\mathfrak{sl}_3$ - skein relations [Kuperberg '96]



(2) boundary $\mathfrak{sl}_3$ - skein relations
[IY]

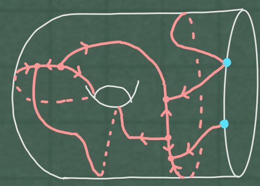


$\rightsquigarrow$ Reidemeister moves



etc.

§. basis, generators, clusters of $\mathcal{S}_{\mathfrak{sl}_3,\Sigma}$



◎ basis webs

$\mathrm{BWeb}_\Sigma = \{$ non-elliptic flat trivalent graphs $\}$

elliptic faces

Theorem [IY] (Confluence theory in [Sikora-Westbury])

BWeb_Σ is a $\mathbb{Z}_A$-basis of $\mathcal{S}_{\mathfrak{sl}_3,\Sigma}$

$\mathbb{Z}_A = \mathbb{Z}[A^{\pm 1}]$

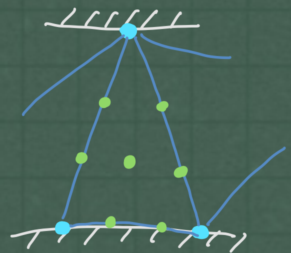
◎ elementary webs EWeb_Σ

$\mathrm{EWeb}_\Sigma \ni G \ :\overset{}{\underset{\mathrm{def}}{\Longleftrightarrow}}\ G \in \mathrm{BWeb}_\Sigma$ & indecomposable by basis webs

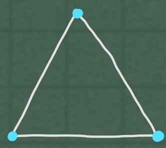
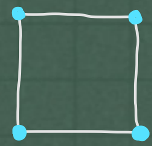
◎ web cluster $\mathcal{C}$

$\mathcal{C}$: an A-commutative subset of EWeb $\rightarrow$ $\mathbb{Z}[\mathcal{C}]$: quantum torus

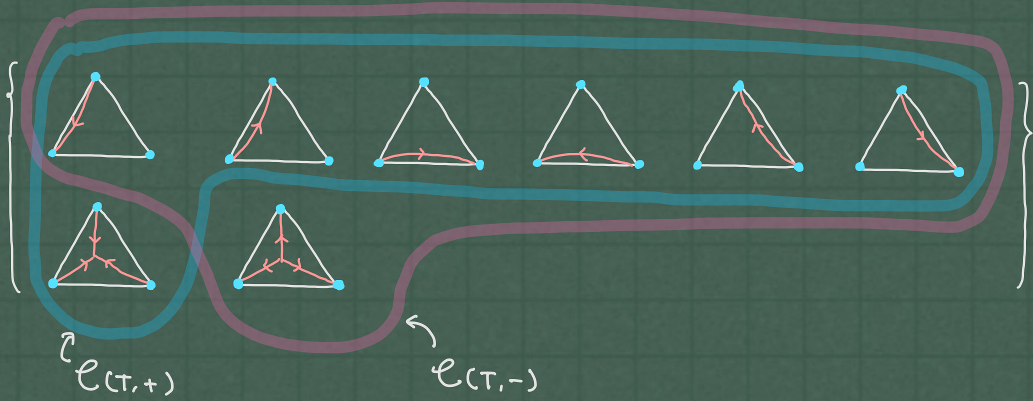
s.t. $\#\mathcal{C} = \#$ vertices of a $\mathfrak{sl}_3$-triangulation



§ $\Sigma = T, Q$

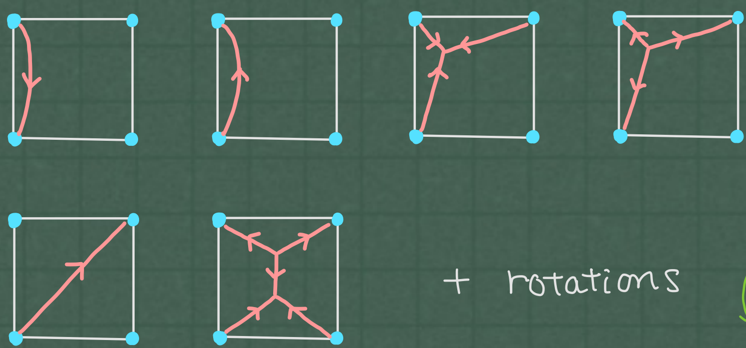
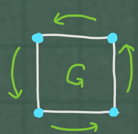
$T :=$  $Q :=$ 

Proposition [IY]

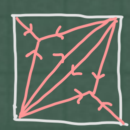
- $\mathrm{EWeb}_T =$ 

- $\mathcal{S}_{T,\mathfrak{sl}_3}$ is generated by EWeb_T

Proposition [IY]

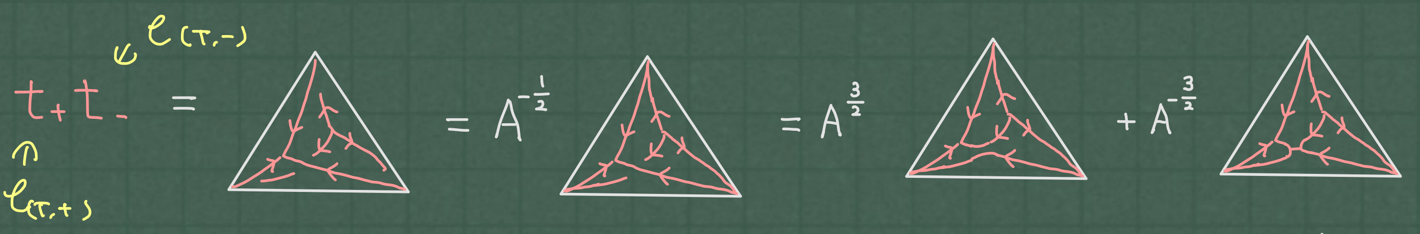
$\mathrm{EWeb}_Q = \{$  + rotations 

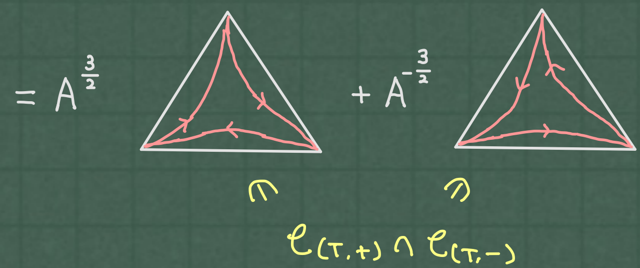
$\}$

$\mathrm{CWeb}_Q = \{ 50$ web clusters $\}$ e.g.  + ori. edges

§ Expansion of webs and positivity

⊗ skein relation : t_+ & t_-



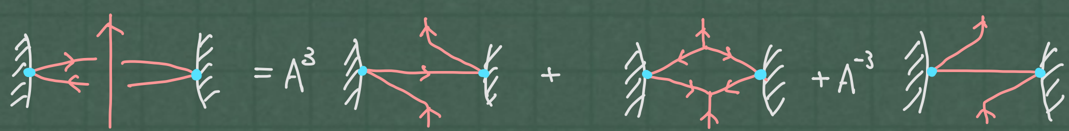


quantum Exchange relation in $\mathcal{A}_{\mathbb{S}_q(T,\mathfrak{sl}_3)}$

$\rightsquigarrow \quad t_- \in \mathbb{Z}_+[e_{(T,+)}, e_{(T,+)}^{-1}] \in \mathcal{U}_{\mathbb{S}_q(\mathfrak{sl}_3,T)}$

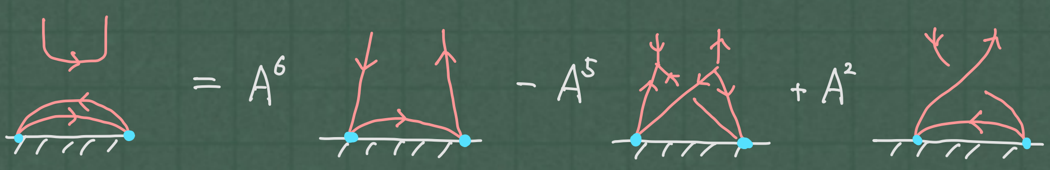
Key Lemma

(1)



$\rightsquigarrow$ positive Laurent expansion in $\mathcal{U}_{\mathbb{S}_q(\mathfrak{sl}_3,\Sigma)}$

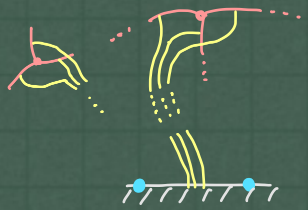
(2)



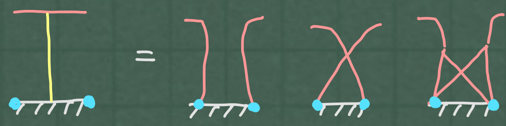
$\rightsquigarrow$ expansion in $\mathcal{A}_{\mathbb{S}_q(\mathfrak{sl}_3,\Sigma)}$

proof of Main theorem

◎ $\mathcal{S}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \hookrightarrow \mathcal{A}_{S_q(\mathfrak{sl}_3,\Sigma)}$ (for simplicity classical case $q = A = 1$)

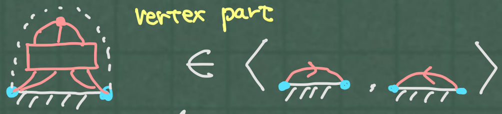


apply (2)



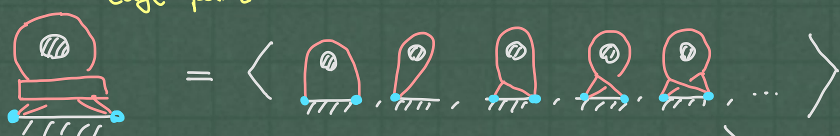
$\rightsquigarrow$ decompose into

vertex part



↖ web in a disk

and (internal) edge part

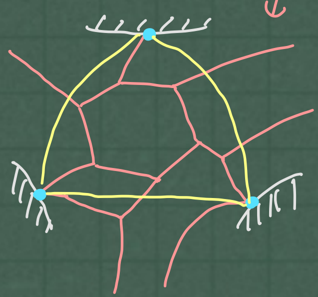


↖ web in a folded quad.

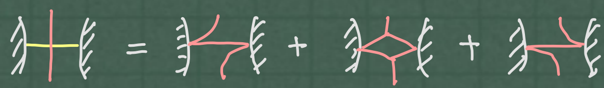
$\rightsquigarrow$ webs in tri. or quad. / boundary webs $\in \mathcal{A}_{S_q(\mathfrak{sl}_3,\Sigma)}$ //

◎ $\mathcal{S}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \hookrightarrow \mathcal{U}_{S_q(\mathfrak{sl}_3,\Sigma)}$

BWeb_Σ

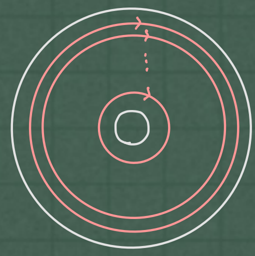


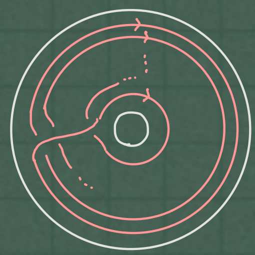
apply (1)



$\rightsquigarrow$ webs in triangles / edges of Δ $\in \mathcal{U}_{S_q(\mathfrak{sl}_3,\Sigma)}$

◎ "quantum positivity"

$$\mathrm{bang}_\Sigma := \left\{ \right.$$  $\left. \subset \mathcal{S}_{\mathfrak{sl}_3,\Sigma} \right\}$

$$\mathrm{brace}_\Sigma := \left\{ \right.$$  $\left. \subset \mathcal{S}_{\mathfrak{sl}_3,\Sigma} \right\}$

Theorem [IY]

$\mathrm{elev}_\Delta = \{$ elevation-preserving webs w.r.t. $\Delta \}$

has positivity for $\mathcal{C}_\Delta$

(Rmk: $\mathrm{elev}_\Delta \supset \mathrm{bang}_\Sigma, \mathrm{brace}_\Sigma$)

◎ elevation-preserving web