Zero stability for the one-row colored $\mathfrak{sl}_3$ Jones polynomial

Main result.

$\{ J^{\mathfrak{sl}_3}_{K,(n,0)}(q) \}$ for "minus-adequate link" K.

then, $\exists\ \mathcal{J}^{\mathfrak{sl}_3}_{K}(q) \in \mathbb{Z}[[q]]$

s.t. $\lim_{n \to \infty} J^{\mathfrak{sl}_3}_{K,(n,0)}(q) = \mathcal{J}^{\mathfrak{sl}_3}_{K}(q)$

Friday Seminar on Knot Theory

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RIMS, Kyoto Univ.

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Colored Jones polynomial

$\{J^{\mathfrak{sl}_2}_{K,n}(q)\}_n$ $\in \mathbb{Q}(q)$ or $\mathbb{C}$ or ...

knots & link. K; n: $(n+1)$-dim. irr. of $\mathfrak{sl}_2$

(a family of invariants of knots & Links)

$J^{\mathfrak{sl}_2}_{K,1}(q)$: the Jones polynomial

$J^{\mathfrak{sl}_2}_{\text{unknot},1}(q) = -[2]$ $\qquad [n] := \dfrac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$

$= -q^{\frac{1}{2}} - q^{-\frac{1}{2}}$

$\Big\downarrow$ for simple Lie alg. $\mathfrak{g}$

+ V_λ : irr.

λ: h.w.

Colored $\mathfrak{g}$ Jones polynomial

$\{J^{\mathfrak{g}}_{K,\lambda}(q)\}_\lambda$

$\mathfrak{g} = \mathfrak{sl}_3$

$\lambda \in \{(n,m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\}$

one-row colors : $\{(n,0) \mid n \in \mathbb{Z}\}$

Properties of the CJP

Theorem [Lê, 2000] $J^{\mathfrak{g}}_{L,\lambda}(q) \in q^{\frac{p}{2}}\mathbb{Z}[q^{\pm 1}]$ (p: determined by $\mathfrak{g}$)

(Integrality)

Define $\delta^*_K(\lambda) := \operatorname{mindeg}(J^{\mathfrak{g}}_{K,\lambda}(q))$ (mindeg: minimum deg.)

$\Big\downarrow$ normalization

$$\hat{J}^{\mathfrak{g}}_{K,\lambda}(q) := \pm q^{-\delta^*_K(\lambda)} J^{\mathfrak{g}}_{K,\lambda}(q) \in \mathbb{Z}[q]$$

$$= \sum_{i=0} a_i q^i \quad (a_0 > 0)$$

Theorem[Garoufalidis-Lê, 2005]

$\{J^{\mathfrak{g}}_{K,\lambda}(q)\}$ is q-holonomic (except G_2)

$\{f_n(q) \in \mathbb{Z}((q))\}_n$ is q-holonomic

if $\exists d \in \mathbb{N}$, $\exists a_\ell \in \mathbb{Z}[u,v]$ $(0 \leq \ell \leq d)$

s.t. $\displaystyle\sum_{\ell=0}^{d} a_\ell(q, q^n) f_{n+\ell}(q) = 0$

$(\forall n \geq 0)$

※ $\{f_n(q)\}_n$: q-holonomic

a, b, c : periodic

$\left(f_n(q) = a_0(n) q^{\delta^*(n)} + a_1(n) q^{\delta^*(n)+1} + \cdots \right)$

$a(n)n^2 + b(n)n + c(n)$

$\shortparallel$ $=: \operatorname{mindeg} f_n(q)$

strong $\Rightarrow$

slope conjecture

- $\delta^*(n)$: a quadratic quasi-polynomial for all but fin. many value of n
- $\forall k \in \mathbb{N}$, $a_k(n)$ satisfies a linear recursion with constant coefficent for all but fin. many value of n

c.f. Generalized Exponential Sum

e.g. $K =$ 

$$f_n(q) = J^{\mathfrak{sl}_2}_{K,n}(q) = \frac{q^{-\frac{1}{2}(n-1)}}{1-q^{-1}} \sum_{k=0}^{n-1} (1-q^{-n})(1-q^{1-n})\cdots(1-q^{k-n})$$

$$\rightsquigarrow f_{n-1}(q) = \frac{q^{n-1} + q^{4-4n} - q^{-n} - q^{1-2n}}{q^{\frac{1}{2}}(q^{n-1} - q^{2-n})} f_n(q)$$

$$+ \frac{q^{4-4n} - q^{3-2n}}{q^{2-n} - q^{n-1}} f_{n+1}(q)$$

e.g. [Garoufalidis - Lê, 2015] coefficients $\{C_K(k)\}$ of the cyclotomic expansion of K.

Theorem[Habiro, 2002]

K : a 0-framed knot

$\exists C_K : \mathbb{Z}_{>0} \to \mathbb{Z}[q^{\pm 1}]$ s.t.

$$J^{\mathfrak{sl}_2}_{K,n}(q) = \sum_{k=1}^{\infty} C_K(k)\, S(n,k)$$

where $\displaystyle S(n,k) = \frac{\prod_{k=n-k+1}^{n+k-1} (q^{\frac{k}{2}} - q^{-\frac{k}{2}})}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$

◎ Stability

Theorem[Dasbach-Lin, 06]

- K : an A - adequate knot

  D : an A - adequate knot diagram

  $G^{red}(D)$: a reduced A-graph of D

$$J^{sl_2}_{K,n}(q) = a_n A^{k_n} + b_n A^{k_n-4} + \cdots + \beta_n A^{l_n+4} + \alpha_n A^{l_n}$$

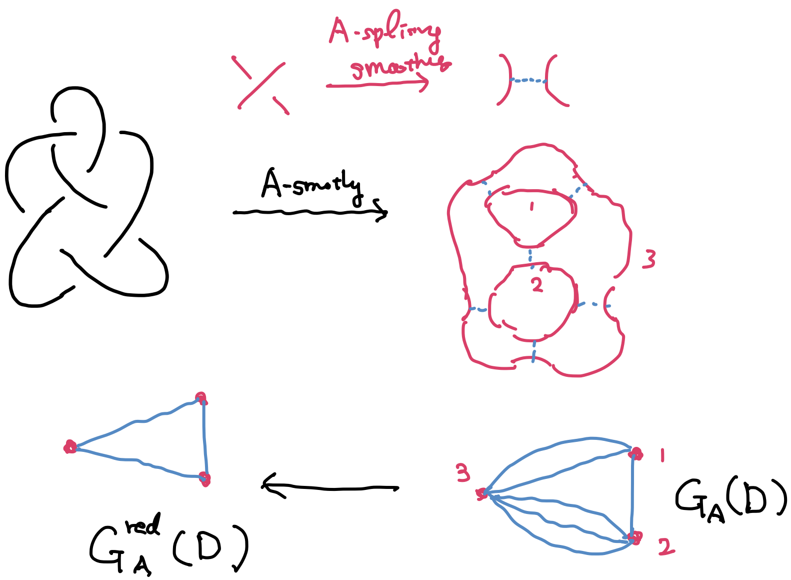
$(q = A^4)$ the highest degree, the lowest degree

some normalization

then, a_n and b_n are determined by $G^{red}(D)$.

e.g.

D =



- Volume-ish theorem for CJP.

  K : an alternating prime, non-torus knot.

  then,

$$2v_0 \cdot (\max(|b_n|, |\beta_n|) - 1) \le \mathrm{Vol}(S^3 - K) \le 10 v_0 \cdot (|b_n| + |\beta_n| - 1)$$

for all n.

v_0 = the volume of an ideal regular hyperbolic tetrahedron.

Theorem[Armond, 2013]

K : an A-adequate link,

then, $^{\exists} J_K^{sl_2}(q) \in \mathbb{Z}[[q]]$ s.t

$$\hat{J}_{K,n}^{sl_2}(q) - J_K^{sl_2}(q) \in q^{n+1}\mathbb{Z}[[q]]$$

tail of $\{\hat{J}_{K,n}^{sl_2}(q)\}$

Theorem[Armond-Dasbach, 2016]

K : an A-adequate link.

D : an A-adequate knot diagram of K.

then $J_K^{sl_2}(q)$ only depends on $G^{red}(D)$

Theorem[Garoufalidis-Lê, 2015]

K : an alternating link.

then $\{\hat{J}_{K,n}^{sl_2}(q)\}$ is k-stable ($^{\forall} k \geq 0$)

- $\{f_n(q) \in \mathbb{Z}[[q]]\}_n$ is k-stable

if $^{\exists}\Phi_0(q), \ldots, {}^{\exists}\Phi_k(q) \in \mathbb{Z}((q))$

s.t. $\lim_{n\to\infty} q^{-k(n+1)}\left(f_n(q) - \sum_{j=0}^{k}\Phi_j(q)\, q^{j(n+1)}\right) = 0$

(※. $\lim_{n\to\infty} f_n(q) = \Phi(q)$

$\underset{def}{\Longleftrightarrow}$ $^{\forall} m \in \mathbb{N}$, $^{\exists} N_m \in \mathbb{N}$

s.t. $f_{N_m}(q) - \Phi(q) \in q^m\mathbb{Z}[[q]]$)

e.g. $N_m = m-1$ if $f_n(q) = \hat{J}^{\mathfrak{sl}_2}_{K,n}(q)$, K: alternating

$$= \sum_{j=0}^{\infty} a_j(n)\, q^j$$

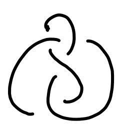
$$\sum_{j=0}^{\infty} a_j q^j$$

0 - stable : $\lim_{n\to\infty} \left(f_n(q) - \Phi_0(q)\right) = 0$, where $\Phi_0(q) = \sum_{j=0}^{\infty} a_j q^j$

$$\Leftrightarrow \quad \underbrace{f_n(q) - \Phi_0(q)}_{\text{tail}} \in q^{n+1}\mathbb{Z}[[q]]$$

$$= (\underbrace{a_{n+1}(n) - a_{n+1}}_{b_0(n)})\, q^{n+1} + (\underbrace{a_{n+2}(n) - a_{n+2}}_{b_1(n)})\, q^{n+2} + \cdots$$

1 - stable : $\lim_{n\to\infty} \left(\underbrace{q^{-(n+1)}\left(f_n(q) - \Phi_0(q)\right)}_{=\sum_{j=0}^{\infty} b_j q^j} - \Phi_1(q) \right) = 0$

K = 

	q^0	q^1	q^2	q^3	…												
Φ_0	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0	-1	…
N=2	1	-1	-1	0	2	0	-2	0	3	0	-3	0	3	0	-3	0	…
N=3	1	-1	-1	0	0	3	-1	-1	-1	-1	5	-1	-2	-2	-1	6	…
N=4	1	-1	-1	0	0	1	2	0	-2	-1	-1	1	3	1	-2	-3	…

Φ_0	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0	-1	…
N=2	0	2	-1	-2	-1	3	0	-3	0	4	0	…					
N=3	0	2	-1	-2	-1	-1	5	-1	-1	-2	-1	…					
N=4	0	2	-1	-2	-1	-1	1	4	1	-2	…						

Φ_1 0 2 -1 -2 -1 -1 1 . . . .

$$\downarrow$$

$$\Phi_2$$

$$\downarrow$$

$$\Phi_3$$

$$\vdots$$

$$\downarrow$$

$$\Phi_k$$

Theorem[Garoufalidis-Vuong, 2017]

K : a trus knot

$\mathfrak{g}$: a rank 2 simple Lie algebra. $\mathfrak{g} = \mathfrak{sl}_3$

then, $\{ J^{\mathfrak{g}}_{K, n\lambda}(q) \}_n$ is c-stable

stable + "cyclotomic"

cyclotomically

Theorem[Y.]

K : a "minus-adequate" oriented link.

$\exists\, J^{\mathfrak{sl}_3}_K(q) \in \mathbb{Z}[[q]]$ s.t.

$$J^{\mathfrak{sl}_3}_{K,(n,0)}(q) - J^{\mathfrak{sl}_3}_K(q) \in q^{n+1}\mathbb{Z}[[q]]$$

i.e. $\{ J^{\mathfrak{sl}_3}_{K,(n,0)}(q) \}$ is zero stable

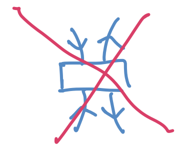
"minus-adequate" oriented link.

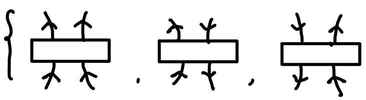
Consider an embeded 4-valent graph into $\mathbb{R}^2$

$G(v_1, v_2, \dots, v_\ell)$

with $\left\{ \begin{array}{l} \text{oriented edges} \\ \text{vertices} = \{v_1, v_2, \dots, v_\ell\} \\ \text{(boxes)} \end{array} \right.$

rectangles

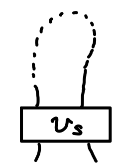


s.t. $v_i \in \{$  $\}$

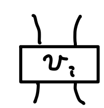
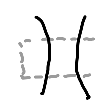
parallel anti-prallel

then $G(v_1, v_2, \dots, v_\ell)$ is adequate

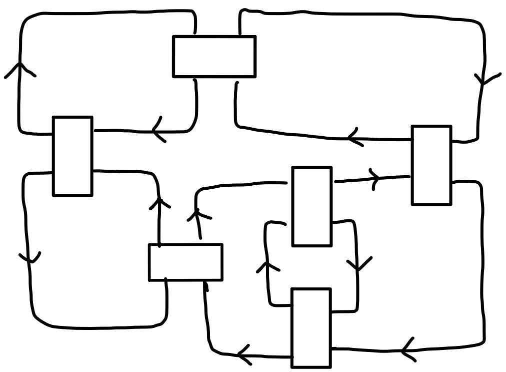
if $\forall s \in \{1, 2, \dots, \ell\}$, there is no arcs

in $G(\bar{v}_1, \dots, \bar{v}_{s-1}, v_s, \bar{v}_{s+1}, \dots, \bar{v}_\ell)$ s.t. 

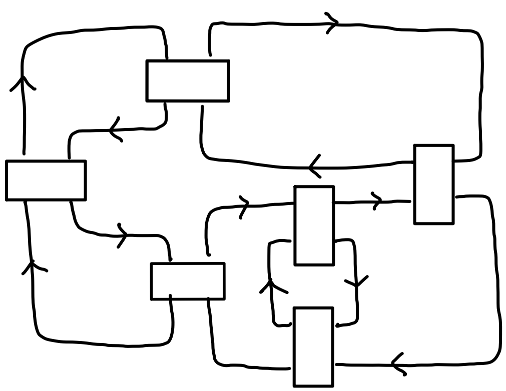
where $\bar{v}_i$ means

replacing  with 

e.g.

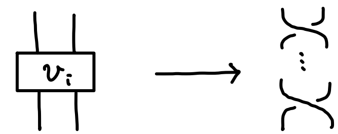


adequate



inadequate

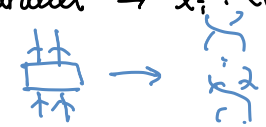
- an oriented link diagram D is **minus-adequate** if $\exists$ an adequate graph $G(v_1, \dots, v_\ell)$ s.t. D is obtained by replacing



$\Big\}$ ℓ_i crossings

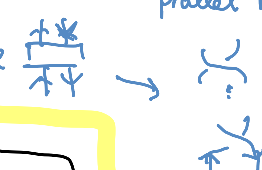
($\asymp$ anti-parallel $\Rightarrow$ ℓ_i : even)

prallel



prallel negative crossing

anti-prallee



antiprallel positive crossing

e.g.