◎広島大学 トポロジー・幾何セミナー

"Skein and cluster algebras of marked surfaces without punctures for $\mathfrak{sl}_3$"

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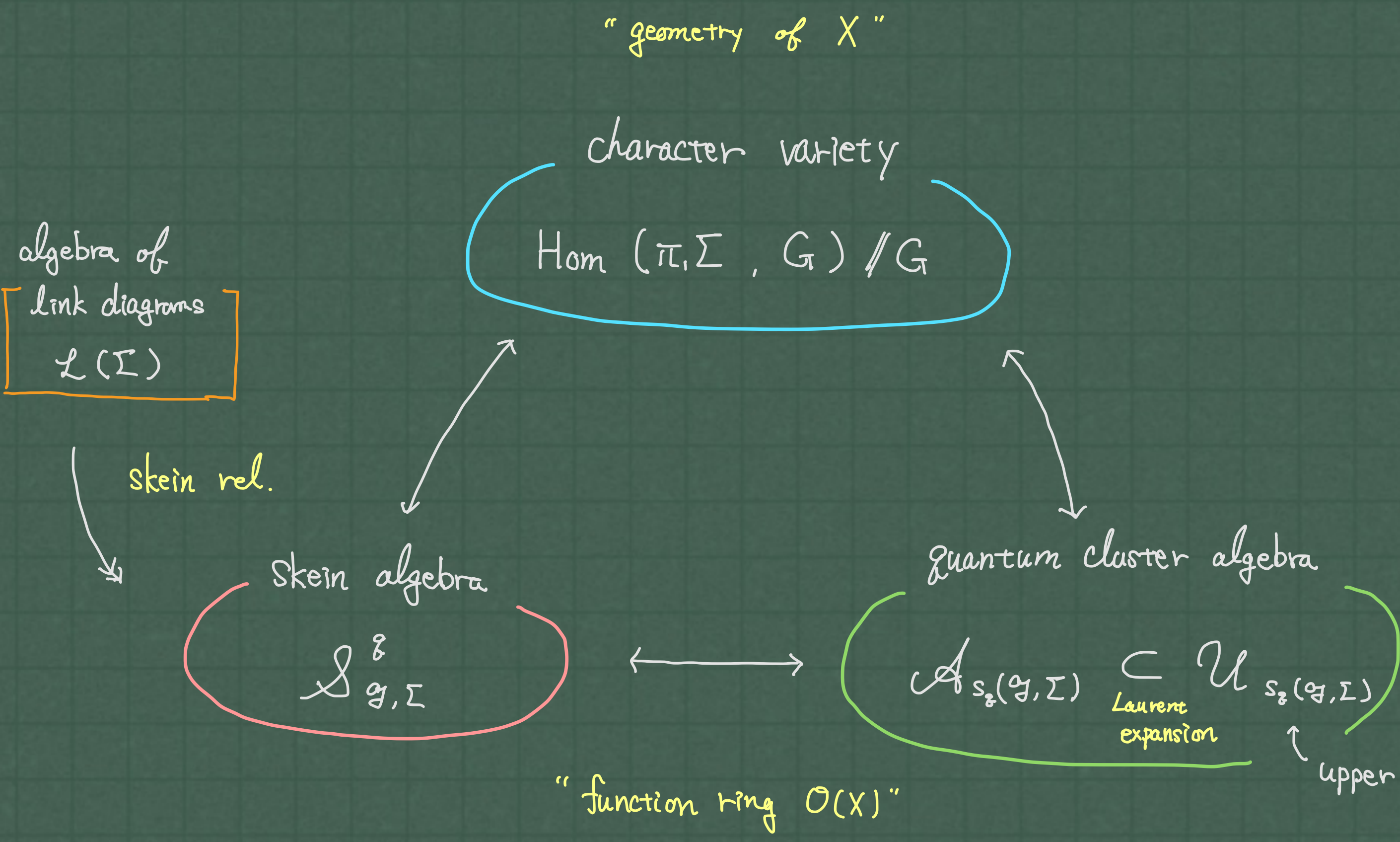
(based on [IY] arXiv : 2101.00643)



skein algebra

- web cluster
- elementary web
- skein relation

$\vdots$

quantum cluster algebra

- cluster ⇝ triangulation of Σ
- cluster variables $\{A_i\}$
- quantum exchange relation

$\vdots$

Conjecture $\mathcal{A}_{S_q(\mathfrak{g},\Sigma)} = \mathcal{S}^q_{\mathfrak{g},\Sigma}[\partial^{-1}] = \mathcal{U}_{S_q(\mathfrak{g},\Sigma)} \subset \mathrm{Frac}\,\mathcal{S}^q_{\mathfrak{g},\Sigma}$

Laurent expansion

$\cup$ $\quad \| \; \mathfrak{g} = \mathfrak{sl}_2$

$\mathcal{A}^q_{\mathfrak{g},\Sigma}$: surface subalgebra

positivity
= coeff. in $\mathbb{Z}_+[q^{\pm 1}]$

Muller (2016) The conjecture is true for $\mathfrak{g} = \mathfrak{sl}_2$

Main result [IY]

- $\mathcal{A}^q_{\mathfrak{sl}_3,\Sigma} \subset \mathcal{S}^q_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \subset \mathcal{U}_{S_q(\mathfrak{sl}_3,\Sigma)} \subset \mathrm{Frac}\,\mathcal{S}^q_{\mathfrak{sl}_3,\Sigma}$
- $\{$elevation preserving webs$\} \subset \mathcal{S}^q_{\mathfrak{sl}_3,\Sigma}$ has positivity.

進展 : $\mathcal{S}^q_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \subset \mathcal{A}_{S_q(\mathfrak{sl}_3,\Sigma)} \subset \mathcal{U}_{S_q(\mathfrak{sl}_3,\Sigma)}$

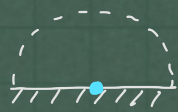
if $\mathcal{U} \subset \mathcal{S} \Rightarrow \mathcal{S} = \mathcal{A} = \mathcal{U}$

§ the $\mathfrak{sl}_3$-skein algebra

marked point

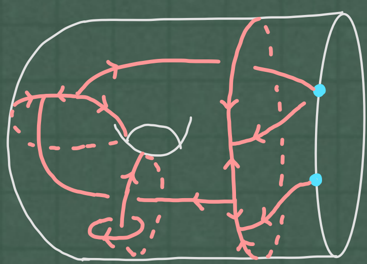
$\mathbb{M} = \{\text{marked pt's}\}$

$\Sigma =$ 

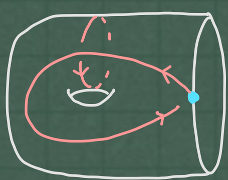
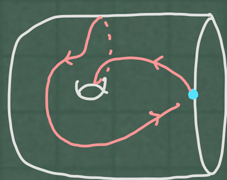
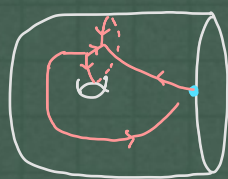


$\rightsquigarrow$

$\mathcal{S}^{A}_{\mathfrak{sl}_3,\Sigma}$: the $\mathfrak{sl}_3$-skein algebra of Σ consisting of $\mathfrak{sl}_3$-webs

e.g. $\mathfrak{sl}_3$-web 

$\mathfrak{sl}_3$-skein relation

 $= A^2$  $+ A^{-1}$ 

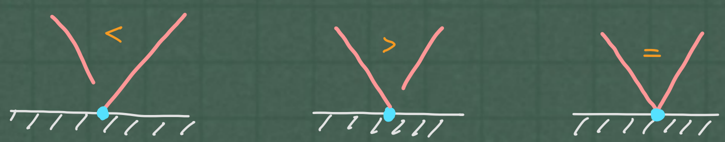
◎ **a tangled trivalent graph on Σ**

$:\Leftrightarrow$ an oriented uni-trivalent graph with

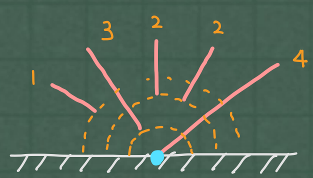
- sink and saurce vertices

- internal crossings on edges

- elevation at a marked point



simultaneous crossing

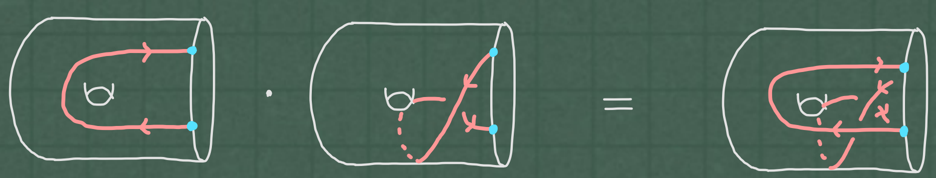


Definition (the $\mathfrak{sl}_3$ - skein algebra of Σ)

$$\mathcal{S}^{A}_{\mathfrak{sl}_3,\Sigma} := \mathrm{span}_{\mathbb{Z}_A} \left\{ \begin{matrix} \text{tangled trivalent} \\ \text{graphs on } \Sigma \end{matrix} \right\} \Big/ \begin{matrix} (1)\ \mathfrak{sl}_3\text{-skein relations} \\ (2)\ \text{boundary } \mathfrak{sl}_3\text{-skein relations} \\ (3)\ \text{isotopy of } \Sigma \text{ rel. to } \partial\Sigma \end{matrix}$$

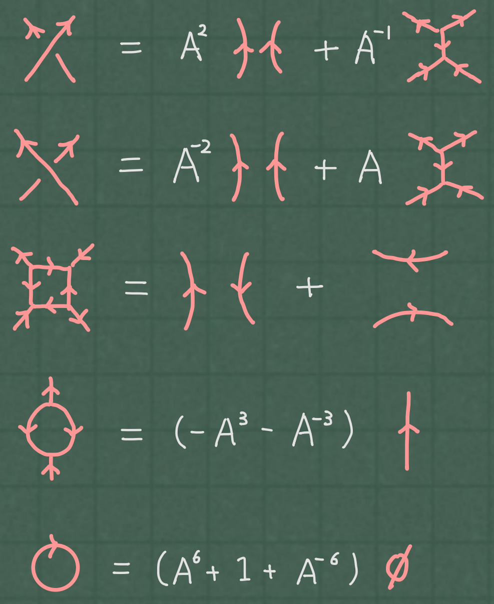
$\mathbb{Z}[A^{\pm\frac{1}{2}}]$ $\quad$ $\mathfrak{sl}_3$-webs

- multiplication of $\mathcal{S}^{A}_{\mathfrak{sl}_3,\Sigma}$

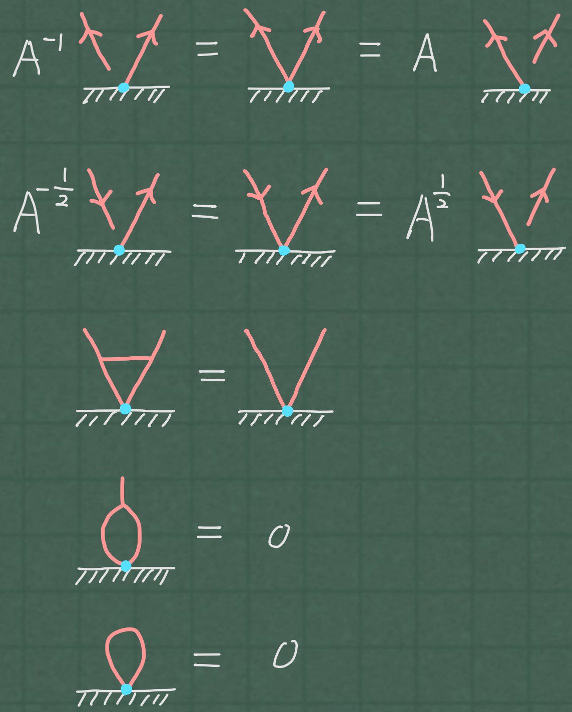


over $\qquad$ under

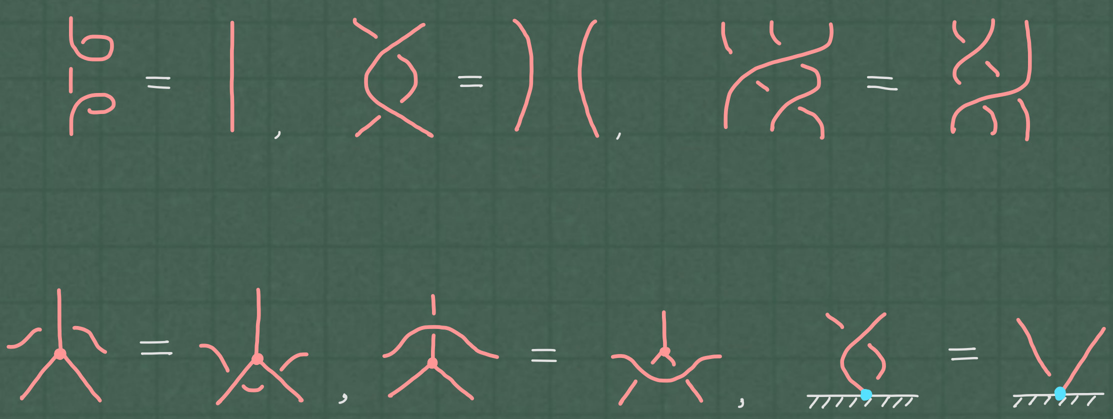
(1) $\mathfrak{sl}_3$ - skein relations [Kuperberg '96]



(2) boundary $\mathfrak{sl}_3$ - skein relations [IY]



$\rightsquigarrow$ (1) (2) realize Reidemeister moves

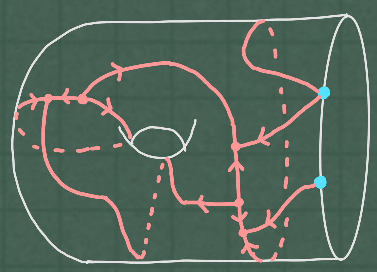


§. basis, generators, clusters of $\mathscr{S}^A_{\mathfrak{sl}_3,\Sigma}$

⊘ basis webs

$$\mathrm{BWeb}_\Sigma = \left\{ \begin{array}{l} \mathfrak{sl}_3\text{-webs represented} \\ \text{by non-elliptic flat trivalent graphs} \end{array} \right\}$$

e.g. non-elliptic flat trivalent graph



elliptic faces

Theorem [IY] (Confluence theory in [Sikora-Westbury])

BWeb_Σ is a $\mathbb{Z}_A$-basis of $\mathscr{S}^A_{\mathfrak{sl}_3,\Sigma}$

$\mathbb{Z}_A = \mathbb{Z}[A^{\pm\frac{1}{2}}]$

◎ elementary webs EWeb_Σ

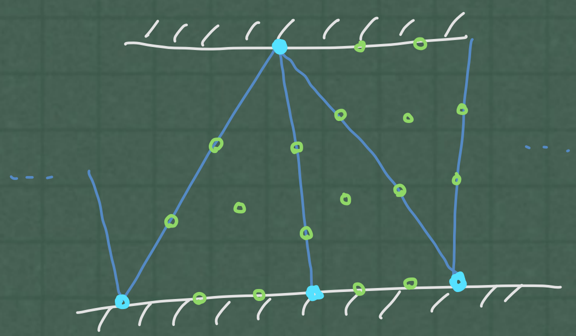
$$\mathrm{EWeb}_\Sigma \ni G \;\underset{\text{def}}{:\Longleftrightarrow}\; G \in \mathrm{BWeb}_\Sigma$$

$$\nexists\, G_1, G_2 \in \mathrm{BWeb} \ \text{ s.t. } \ G = A^{\circledast} G_1 G_2$$

◎ web cluster $\mathscr{C}$

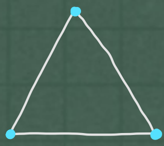
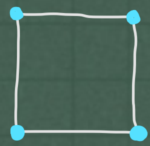
an A-commutative subset of EWeb

s.t. $\#\mathscr{C} = \#$ vertices of a $\mathfrak{sl}_3$-triangulation

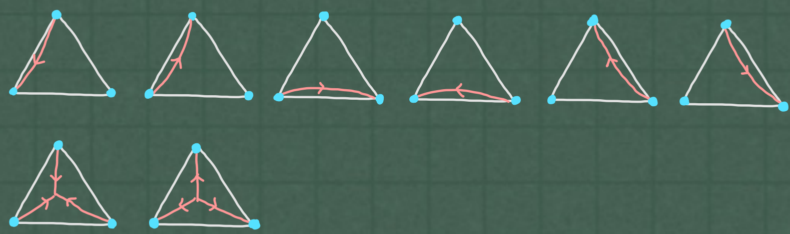


※ $$\begin{array}{l} \mathscr{C} \subset \mathrm{EWeb}_\Sigma \subset \mathrm{BWeb}_\Sigma \subset \mathscr{S}^A_{\mathfrak{sl}_3,\Sigma} \\ \cap \\ \mathrm{CWeb}_\Sigma \end{array}$$

§ $\Sigma = T, Q$

$T :=$  $Q :=$ 

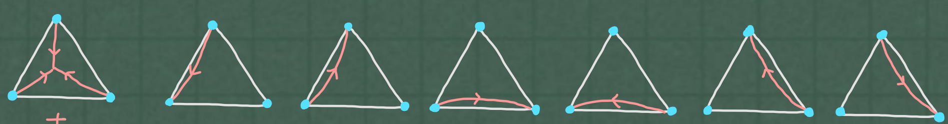
Proposition [IY]

- $\mathrm{EWeb}_T = \{$  $\}$

- EWeb_T generates $\mathscr{S}^{A}_{\mathfrak{sl}_3, \cancel{\Sigma}\, T}$

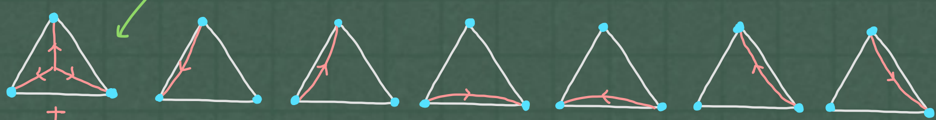
the set of web clusters

Proposition [IY] $\mathrm{CWeb}_T = \{ \mathscr{C}_{(T,+)}, \mathscr{C}_{(T,-)} \}$

$\mathscr{C}_{(T,+)} = \{$  $\}$

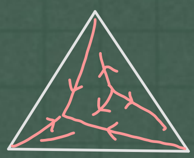
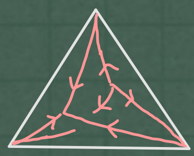
t_+

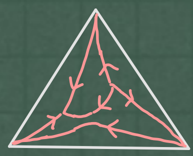
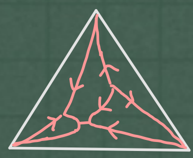
non-commutative

$\mathscr{C}_{(T,-)} = \{$  $\}$

t_-

● skein relation : t_+ & t_-

$t_+ t_-$ =  $= A^{-\frac{1}{2}}$ 

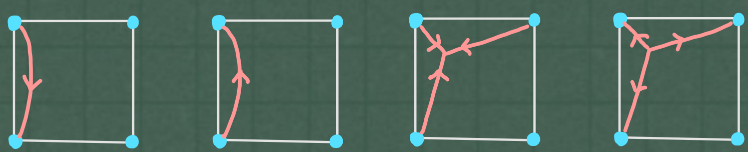
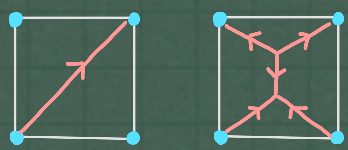
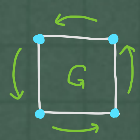
$= A^{\frac{3}{2}}$  $+ A^{-\frac{3}{2}}$ 

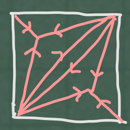
$= A^{\frac{3}{2}}$  $+ A^{-\frac{3}{2}}$ 

"quantum Exchange relation"

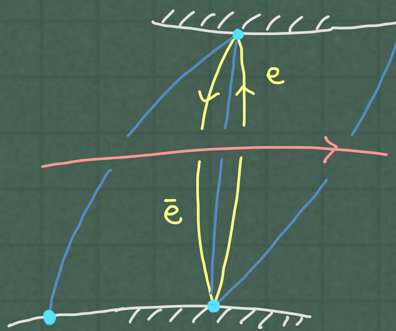
in $\mathcal{A}_{S_q(T,\mathfrak{sl}_3)}$

Proposition [IY]

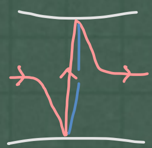
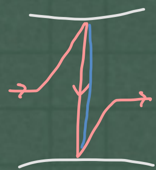
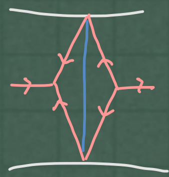
$\mathrm{EWeb}_Q = \Big\{$   + rotations  $\Big\}$

$\mathrm{CWeb}_Q = \{$ 50 web clusters $\}$ e.g.  + ori. edges

§ Expansion of webs and positivity



$w \in \mathscr{S}^A_{\Sigma,\mathfrak{sl}_3}$

$w\, e\bar{e} \;=\; A^{\circ}$  $+\; A^{\triangle}$  $+\; A^{\square}$ 

Theorem [IY]

Δ : an ideal triangulation of Σ.

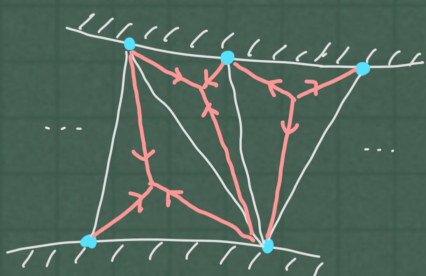
$t(\Delta)$: triangles in Δ

a web cluster of T with t_ε

$\forall\, G \in \mathrm{BWeb}$, $\exists\, J_G$: monomial in $\mathcal{C}_\Delta := \bigcup_{T\in t(\Delta)} \mathcal{C}_{T_\varepsilon}$

s.t. $G\cdot J_G \in \langle \mathcal{C}_\Delta \rangle_{alg} \subset \mathscr{S}^A_{\mathfrak{sl}_3,\Sigma}$

e.g.

$\mathcal{C}_\Delta =$  $+$ ori. edges

$\rightsquigarrow$ • Any $\mathfrak{sl}_3$-web has a Laurent expression in $\mathcal{C}_\Delta$

◎ "quantum positivity"

S : a certain subset of $\mathfrak{sl}_3$-webs

S has quantum positivity w.r.t. Δ

$\underset{\text{def}}{\Longleftrightarrow}$ $^\forall G \in S$ has a Laurent expression in $\langle \mathcal{C}_\Delta \rangle_{\mathrm{alg}}$ with coefficients in $\mathbb{Z}_+[q^{\pm 1}]$

$\mathrm{bang}_\Sigma := \Bigg\{$  $\underset{\text{emb. along a SCC in } \Sigma}{\subset} \mathcal{S}^A_{\Sigma, \mathfrak{sl}_3} \Bigg\}$

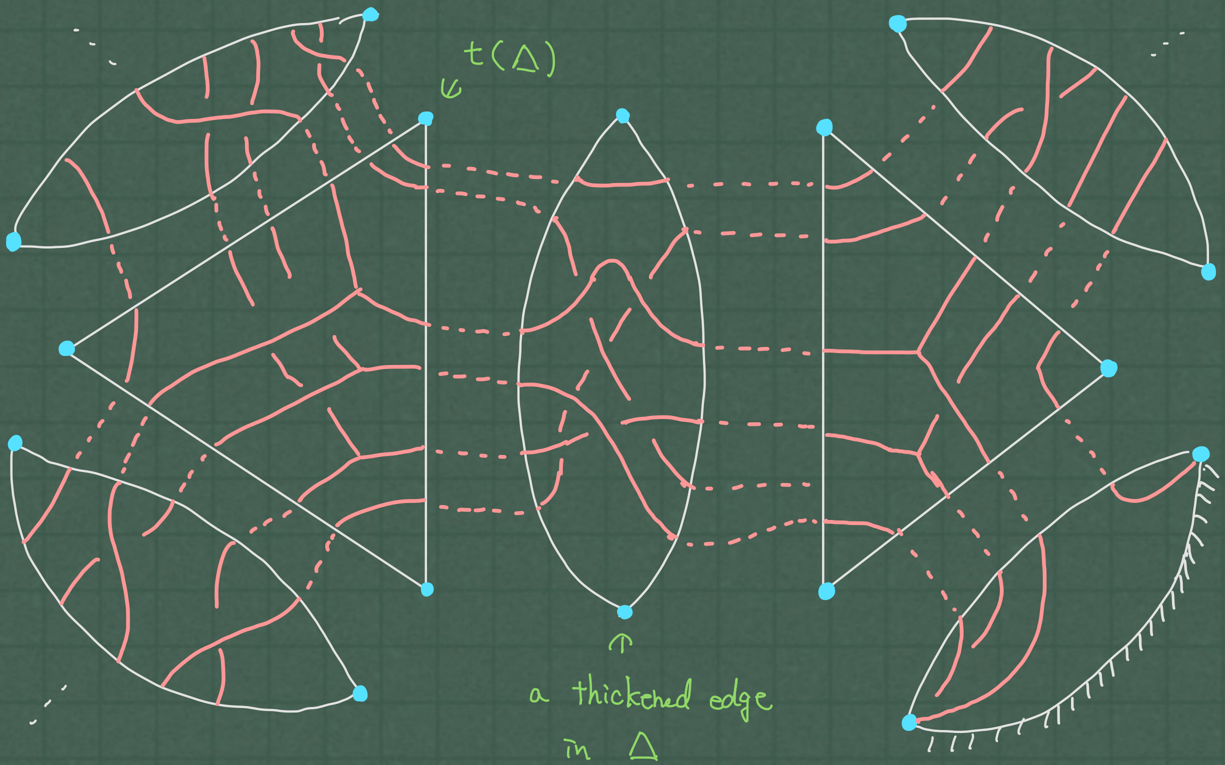
$\mathrm{brace}_\Sigma := \Bigg\{$  $\underset{\text{emb. along a SCC in } \Sigma}{\subset} \mathcal{S}^A_{\Sigma, \mathfrak{sl}_3} \Bigg\}$

Theorem [IY]

$\mathrm{elev}_{\Delta} = \{ \text{elevation-preserving webs w.r.t. } \Delta \}$

has positivity for $\mathcal{C}_{\Delta}$

◎ elevation-preserving web



<u>Remark</u> $\mathrm{elev}_{\Delta} \supset \mathrm{bang}_{\Sigma} \cup \mathrm{brace}_{\Sigma}$