Infinite Analysis 21 Workshop
Around Cluster Algebras
2021-09-28

Skein realization of cluster algebras with coefficients from marked surfaces

Wataru Yuasa (RIMS JSPS research fellow)

with Tukasa Ishibashi (RIMS)
Shunsuke Kano (Tohoku Univ.)

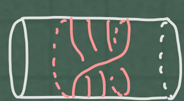
§ Introduction

today Σ = "walled" surface with coefficient

- For a "surface" Σ, show "$\cong$"

$$
\begin{array}{ccc}
\text{algebra of knots in } \Sigma \times [0,1] & & \\
\big\downarrow \text{ skein rel.} & & \\
\text{Skein algebra} & \cong & \text{quantum cluster algebra} \\
\big\downarrow\ q \to 1 & & \big\downarrow \ \big\uparrow \text{ compatibility matrix} \\
\text{Skein alg. at } q = 1 & \cong & \text{Cluster algebra}
\end{array}
$$

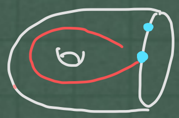
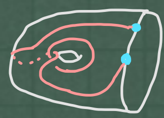
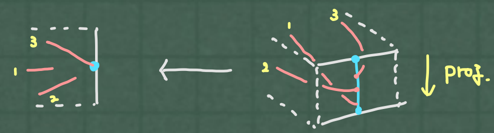
"trace of line op." ↖ $\mathrm{Hom}(\pi_1, G) /\!/ G$ ↗ cluster coordinates.

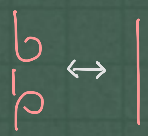
- Application
  - positive elements e.g  bracelet $\longleftrightarrow$ theta basis
  - expansion formulae

$\Sigma :=$ 

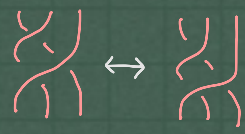
, triangulable

$\mathrm{Tang}(\Sigma) := \{$ tangle diagrams on $\Sigma \}$ / (R1′) (R2) (R3) (mR)

e.g. $T_1 =$  $T_2 =$  

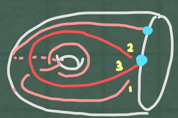
(R1′) 

(R2) 

(R3) 

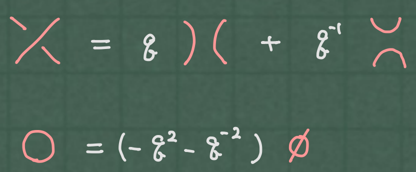
(mR) 

◎ $\mathbb{Z}_q \mathrm{Tang}(\Sigma)$: the algebra of tangles on Σ

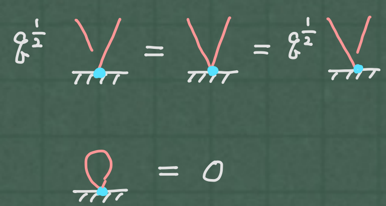
$T_1 \cdot T_2 :=$ 

◎ Skein algebra $\mathcal{S}(\Sigma) := \mathbb{Z}_q \mathrm{Tang}(\Sigma) /$ "skein relation"

skein relation



} the Kauffman bracket skein rel.



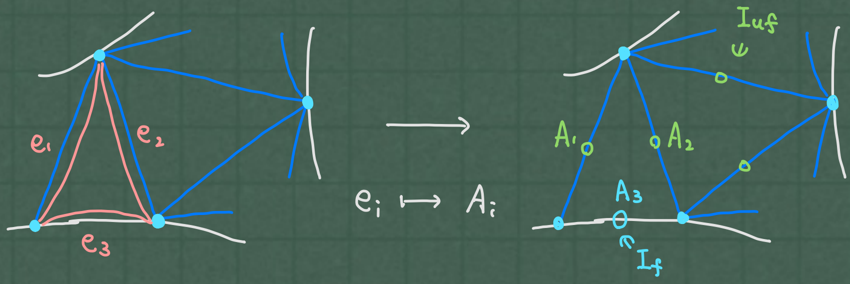
} the boundary skein rel.

↑ defined by Muller

Thm (Muller 2016)

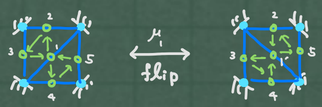
- $\mathcal{A}_{S_q(\mathfrak{sl}_2,\Sigma)} \subset \mathcal{S}(\Sigma)[\partial^{-1}] \subset \mathcal{U}_{S_q(\mathfrak{sl}_2,\Sigma)} \subset \mathrm{Frac}\,\mathcal{S}(\Sigma)$
- $\mathcal{A}_{S_q(\mathfrak{sl}_2,\Sigma)} = \mathcal{U}_{S_q(\mathfrak{sl}_2,\Sigma)}$

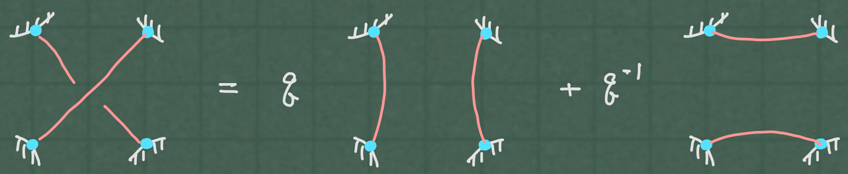
$$\Rightarrow \mathcal{A}_{S_q(\mathfrak{sl}_2,\Sigma)} = \mathcal{S}(\Sigma)[\partial^{-1}] = \mathcal{U}_{S_q(\mathfrak{sl}_2,\Sigma)}$$



$e_i \longmapsto A_i$

e.g. Exchange relation





skein rel.

exch. rel.

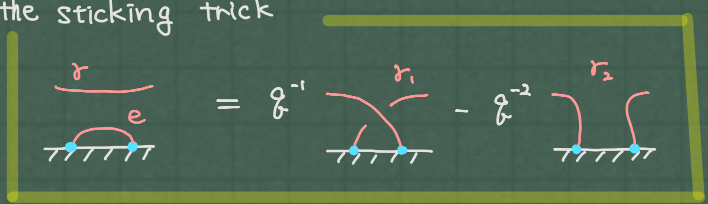
$$A_1 A_{1'} = q A_3 A_5 + q^{-1} A_2 A_4$$

c.f. [Ishibashi - Y.] $\mathcal{S}_{\mathfrak{g}}(\Sigma)[\partial^{-1}] \subset \mathcal{A}_{S_q(\mathfrak{g},\Sigma)}$ $\quad \mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sp}_4$

Two expansions

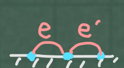
① $\mathcal{S}(\Sigma)[\partial^{-1}] \longrightarrow \mathcal{A}_{S_q(\mathfrak{sl}_2,\Sigma)}$

the sticking trick



$$\gamma e = q^{-1}\gamma_1 - q^{-2}\gamma_2$$

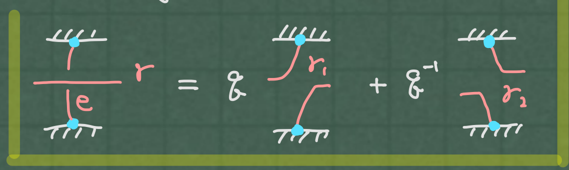
$$\rightsquigarrow \gamma e^{n_1} e'^{n_2} = \sum a_i(q) f_i$$



f_i ≈ monomial in simple arcs = monomial in cluster var.'s

② $\mathcal{S}(\Sigma)[\Delta^{-1}] \underset{\text{"positive"}}{\longrightarrow} \mathcal{U}_{S_q(\mathfrak{sl}_2,\Sigma)}$

the cutting trick



$\gamma e = q\gamma_1 + q^{-1}\gamma_2$

$\rightsquigarrow \gamma\left(\prod_{i \in I_{uf}(\Delta)} e_i^{n_i}\right) = \sum b_i(q)\, g_i$

monomial in edges of Δ = cluster monomial in $\mathcal{C}_\Delta$

eg. γ : a bracelet $\Rightarrow b_i(q) \in \mathbb{Z}_+[q^{\pm\frac{1}{2}}]$

Q. with coefficients case ?

(Muller at $q=1$ is the coefficient-free case)

- Define a skein algebra "with coefficients"
- Construct an isomorphism.

§ The skein algebra of a walled surface

◎ walled surface $\Sigma_{\mathbb{W}}$

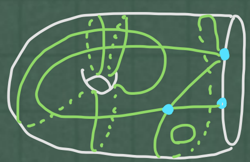
Σ : a surface with marked points
$M = M_\partial \sqcup P$

a wall $\mathbb{W} = \mathbb{W}_\infty \sqcup \mathbb{W}_0$ of Σ

: a collection of $\underbrace{\text{simple loops}}_{\mathbb{W}_0}$ & $\underbrace{\text{simple arcs}}_{\mathbb{W}_\infty}$

s.t. $\begin{cases} \cup \mathbb{W} \text{ only has transverse double points} \\ \partial \xi \in M \quad \text{for } \forall \xi \in \mathbb{W}_\infty \end{cases}$

a chamber of $\Sigma_{\mathbb{W}}$: a connected component of $\Sigma \setminus \cup \mathbb{W}$

e.g. • $\Sigma_{\mathbb{W}} =$ 

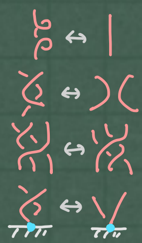
$\mathbb{W}$

• $\mathbb{W}$ = an ideal triangulation $\longleftrightarrow$ principal coefficients

◎ tangles in $\Sigma_{\mathbb{W}}$: $\mathrm{Tang}(\Sigma_{\mathbb{W}})$

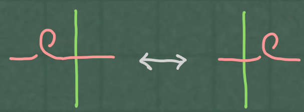
a tangle diagram on $\Sigma_{\mathbb{W}}$: a tangle diagram on Σ
which intersects with $\cup \mathbb{W}$ transversally

$\mathrm{Tang}(\Sigma_{\mathbb{W}}) := \{\text{tangle diag. on } \Sigma_{\mathbb{W}}\} / $ (R1′) (R2) (R3) (mR) in a chamber
(wR) (wwR) (kink)

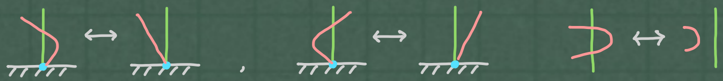


(wR) 

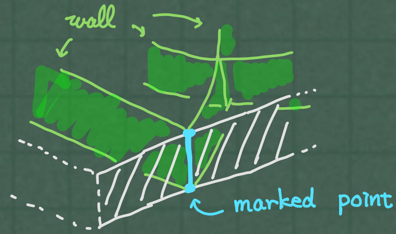
(wwR) 

(kink) 

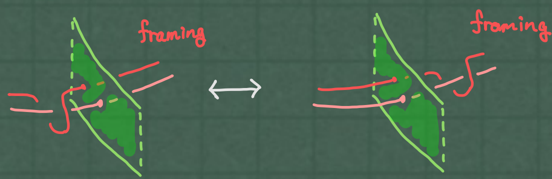
<u>Remark</u> We forbid the following moves :



☆ In $\Sigma \times [0,1]$



(wR) 

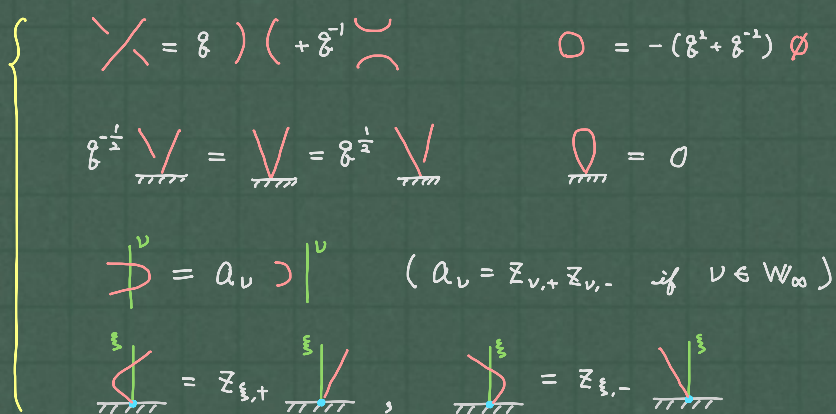
(kink) 

◎ The skein algebra $\mathcal{S}(\Sigma_W)$ of Σ_W $(P = \emptyset)$

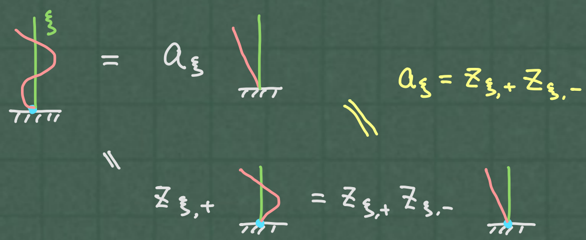
$$R_W := \mathbb{Z}[q^{\pm\frac{1}{2}}, z_{\xi,+}^{\pm1}, z_{\xi,-}^{\pm1}, a_\zeta^{\pm1} \mid \xi \in W_\infty, \zeta \in W_0]$$

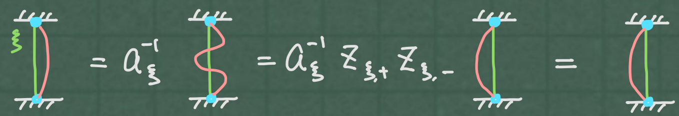
$$\mathcal{S}(\Sigma_W) = R_W \,\mathrm{Tang}(\Sigma_W) \,/\, \underline{\text{skein relations}}$$

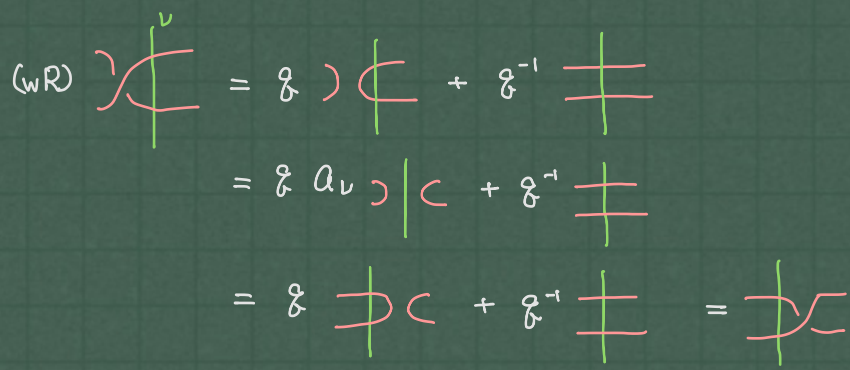
skein rel.

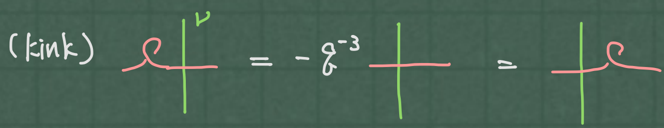


e.g.









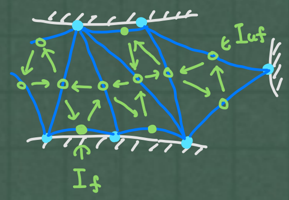
<u>Proposition</u>

- $\mathcal{S}(\Sigma_W)$ is an R_W-free module
- $\{\text{multi-curves}\}/$  is a basis of $\mathcal{S}(\Sigma_W)$
- simple arcs are non-zero divisors.
- $\mathcal{S}(\Sigma_W) \subset \mathrm{Frac}\, \mathcal{S}(\Sigma_W)$ (if W is an ideal triangulation)

§ Integral $\mathcal{X}$-laminations & Coefficients

Δ : an ideal triangulation

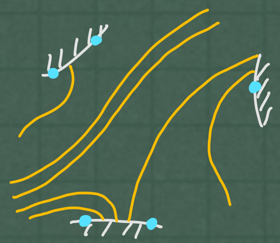
$I = I_f \sqcup I_{uf} \longleftrightarrow e(\Delta)$: the set of edges of Δ



- **Integral $\mathcal{X}$-lamination L**

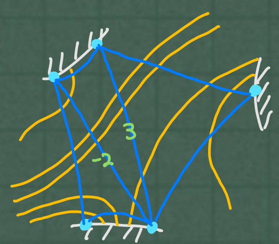
$$L = \bigsqcup_i \gamma_i \Big/ \sim_{\substack{\text{isotopy} \\ \text{rel. to } M_\partial}}$$

s.t.
- γ_i is a simple loop or simple arc $\partial\gamma_i \in \partial\Sigma \setminus M_\partial$
- γ_i does not bound a disk
- $\{\gamma_i\}$: mutually disjoint curves

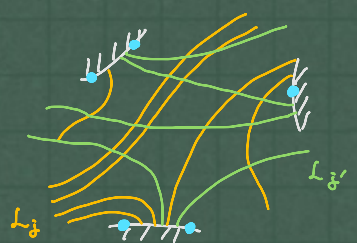


– the coordinate $x^\Delta = (x_i^\Delta)_{i \in I_{uf}}$ of L

$$x_i^\Delta(L) := \sum_j \widehat{\mathrm{Int}}_\Delta(e_i, \gamma_j)$$

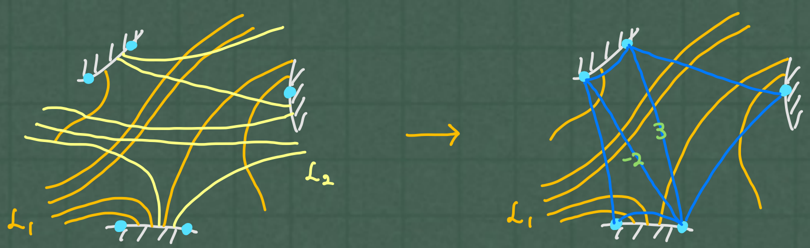


– a multi lamination $\mathbb{L} = (L_j)_{j \in J}$ is a tuple of integral $\mathcal{X}$-laminations.



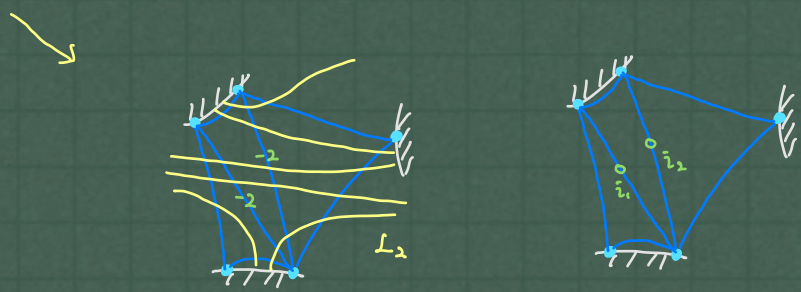
⊛ coefficients $p^{\triangle}_{\mathbb{L}} = (p^{\triangle}_{\mathbb{L},i})_{i\in I}$ $\quad p^{\triangle}_{\mathbb{L},i} \in \mathrm{Trop}(u_j \mid j \in J)$

$$p^{\triangle}_{\mathbb{L},i} := \begin{cases} \prod_{j\in J} u_j^{x_i^{\triangle}(L_j)} & \text{if } i \in I_{uf} \\ 1 & \text{if } i \in I_f \end{cases}$$



$p^{\triangle}_{\mathbb{L},i_1} = u_1^{-2} u_2^{-2}$

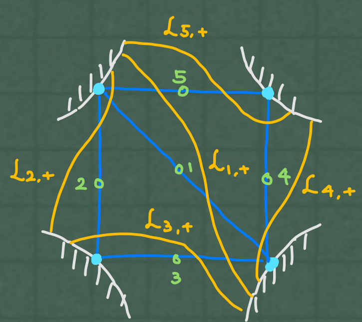
$p^{\triangle}_{\mathbb{L},i_2} = u_1^{3} u_2^{-2}$



e.g. (Principal coefficients)

$\mathbb{P} = \mathrm{Trop}(u_i \mid i \in I)$

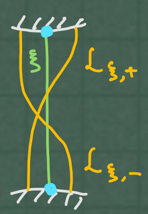
$\mathbb{L}^+(\triangle) := (L_{i,+} \mid i \in I)$



e.g. (Coefficients associated with a wall W)

$\mathbb{P} = \mathrm{Trop}(u_{\xi,\pm}, u_{\eta} \mid \xi \in W_{\infty}, \eta \in W_0)$

$\mathbb{L}(W) := (L_{\xi,\pm}, L_{\eta} \mid \xi \in W_{\infty}, \eta \in W_0)$



§ Main theorem & Examples

shear coord. of $\mathbb{L}$

$$\widetilde{B}_\Delta = \left(B_\Delta \,\middle|\, \boxed{\;|\;|\;\cdots\;} \right)$$

Theorem • For a walled surface Σ_W.

$$\mathscr{S}(\Sigma_W)[\partial^{-1}]\big|_{q=1} \cong \mathscr{A}_\Sigma[\mathbb{L}(W)]$$

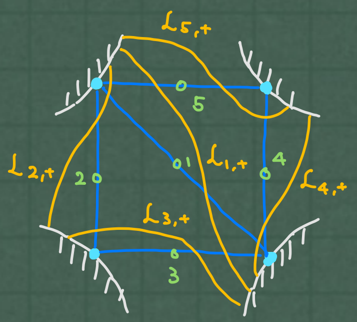
• For a multi-lamination $\mathbb{L} = (\mathcal{L}_j)_{j\in J}$

$$\mathscr{S}(\Sigma_{W^+(\mathbb{L})})[\partial^{-1}]\Bigg|_{\substack{q=1,\ z_{\xi,-}=1 \\ z_{\xi,+} = z_j\ (\xi\in\mathcal{L}_j) \\ a_\eta = z_j\ (\eta\in\mathcal{L}_j)}} \cong \mathscr{A}_\Sigma(\mathbb{L})$$

sketch of proof : $\begin{cases} \mathscr{S}[\partial^{-1}] \longrightarrow \mathscr{A}, & \text{expansion by the sticking trick} \\ \mathscr{A} \longrightarrow \mathscr{S}[\partial^{-1}], & A_i \mapsto e_i \\ \text{Compare skein \& exchange relations} \end{cases}$

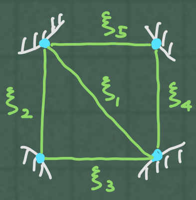
◎ Exchange & skein relations (examples)

◎ $A_k A_k' = P_k^+ \prod_{j\in I_{uf}} A_j^{[b_{kj}]_+} + P_k^- \prod_{j\in I_{uf}} A_j^{[-b_{kj}]_+}$ ◎ $P_{\mathbb{L},i} = \prod_{j\in J} u_j^{x_i^\Delta(\mathcal{L}_j)}$



$\mathbb{L}^+(\Delta)$

$\longrightarrow$

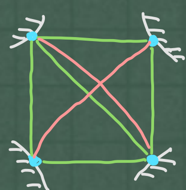
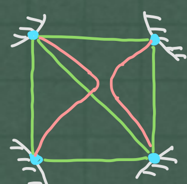
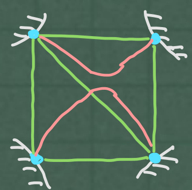


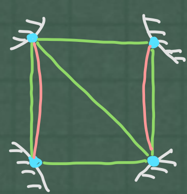
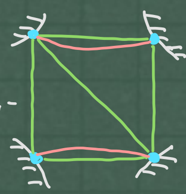
$W^+(\mathbb{L}^+(\Delta))$

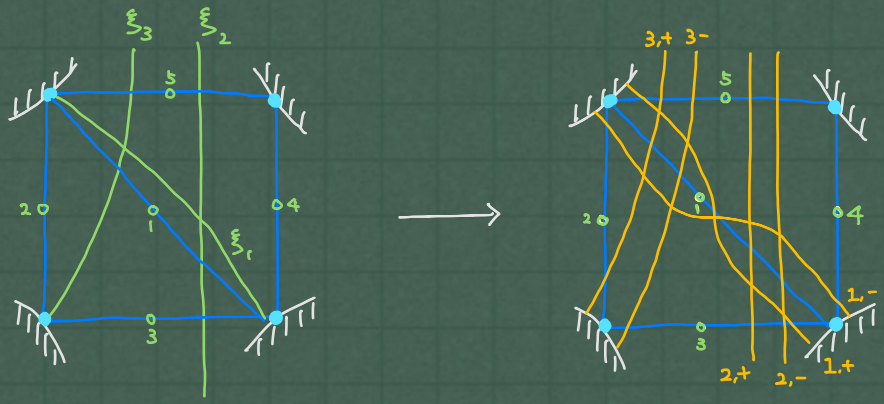
$x_1^\Delta(\mathcal{L}_{1,+}) = +1$

$x_i^\Delta(\mathcal{L}_{i,+}) = 0 \quad (i\neq 1)$

$A_1 A_1' = u_1 A_2 A_4 + A_3 A_5$

 =  + 

$= z_{1,+}$  $+ \underset{\substack{\| \\ 1}}{z_{1,-}}$ 

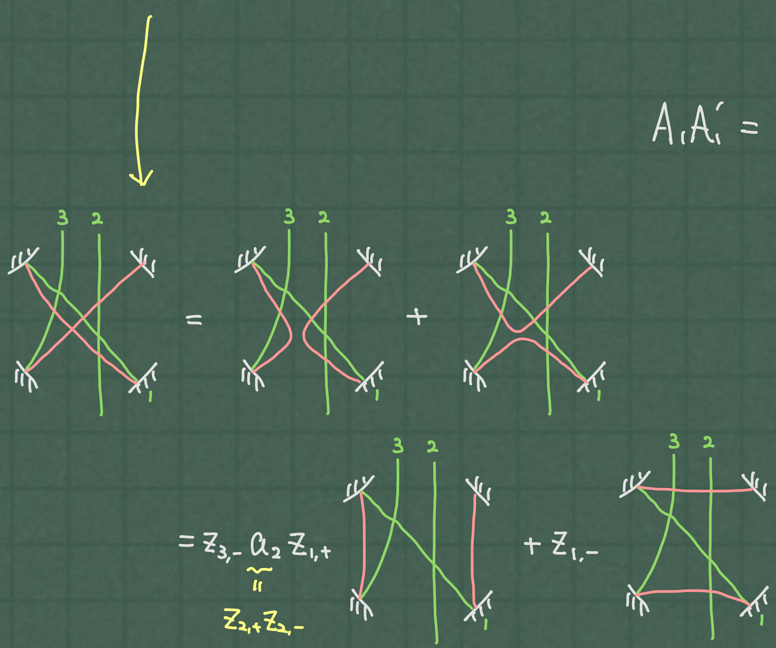


W $\qquad\qquad\qquad$ $\mathbb{L}(W)$

$$x^{\triangle}_{\alpha_1}(\mathcal{L}_{1,+}) = +1$$
$$x^{\triangle}_{\alpha_1}(\mathcal{L}_{1,-}) = -1$$
$$x^{\triangle}_{\alpha_1}(\mathcal{L}_{2,+}) = +1$$
$$x^{\triangle}_{\alpha_1}(\mathcal{L}_{2,-}) = +1$$
$$x^{\triangle}_{\alpha_1}(\mathcal{L}_{3,-}) = +1$$
$$x^{\triangle}_{\alpha_1}(\mathcal{L}_{3,+}) = 0$$

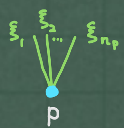
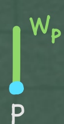
$$P^{+}_{\mathbb{L},1} = u_{1,+}u_{2,+}u_{2,-}u_{3,-}$$
$$P^{-}_{\mathbb{L},1} = u_{1,-}$$

$$A_1A_1' = u_{1,+}u_{2,+}u_{2,-}u_{3,-}A_2A_4 + u_{1,-}A_3A_5$$



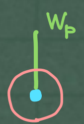
$$e_1e_1' = z_{1,+}z_{2,+}z_{2,-}z_{3,-}\,e_2e_4 + z_{1,-}\,e_3e_5$$

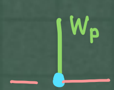
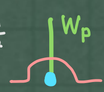
§ punctured case (in progress)

For a puncture  =  $\quad z_p := \boxed{\prod_i z_{\xi_i,+} = \prod_i z_{\xi_i,-}}$

condition coming from  = 

- skein relation

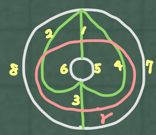
 $= (q + q^{-1})\, z_p$  ,

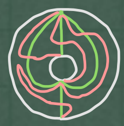
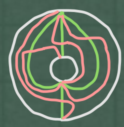
$\underbrace{v_p}$  $= q^{\frac{1}{2}} z_p$  $+\ q^{-\frac{1}{2}}$ 

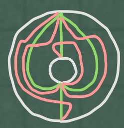
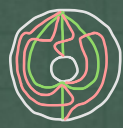
↑ new variable (↔ tag)

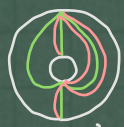
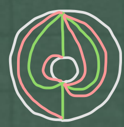
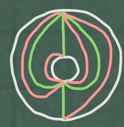
§ Examples of Laurent expansions

(i) Musiker - Williams (2013) "Matrix formulae and Skein ... "

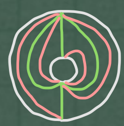
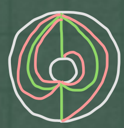
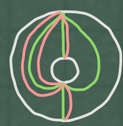
$A_1A_2A_3A_4$ 

$=$  $+$  $+$ 

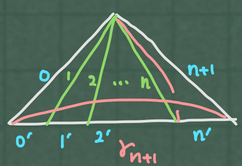
$+$  $+$  $+$ 

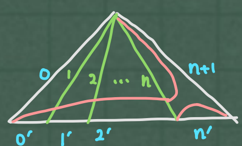
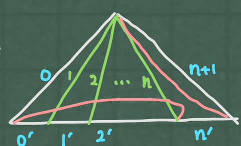
$= Z_{1234,-}$  $+ Z_{134,-} Z_{2,+}$  $+ Z_{14,-} Z_{23,+}$ 

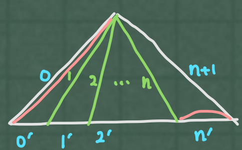
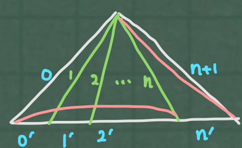
$A_1A_3A_4^2$ $\quad$ $A_2^2A_6A_8$ $\quad$ $A_2A_4A_6A_7$

$+ Z_{34,-} Z_{12,+}$  $+ Z_{4,-} Z_{123,+}$  $+ Z_{1234,-}$ 

$A_2A_4A_5A_8$ $\quad$ $A_2^2A_5A_7$ $\quad$ $A_1A_2^2A_3$

(ii) 

$= q$  $+ q^{-1}$ 

$= q Z_{+,12\cdots n}$  $+ q^{-1} Z_{-,n}$ 

$$\therefore \ \gamma_{n+1} A_n = q Z_{+,1\cdots n} A_0 A_{n'} + q^{-1} Z_{-,n} \alpha(n) A_{n+1}$$

$$\rightsquigarrow \ \gamma_{n+1} = \sum_{k=0}^{n} Z_{+,1\cdots n-k} Z_{-,n-k+1\cdots n} A_{n-k}^{-1} A_{n-k+1}^{-1} A_{n+1} A'_{n-k} A_0$$