2022-07-26
The 13th KOOK-TAPU

State-Clasp correspondence for skein algebras

a joint work with Tsukasa Ishibashi (Tohoku Univ.)

Wataru Yuasa (OCAMI / RIMS)

§0 Introduction

$(\Sigma, \mathbb{M}, \mathbb{B})$: an unpunctured marked surface with marked points $\mathbb{M}$ and boundary intervals $\mathbb{B}$

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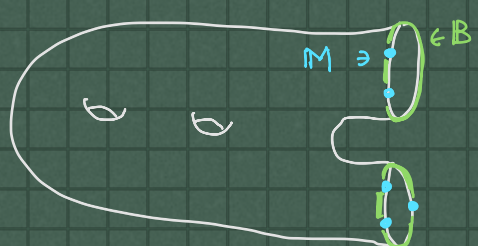


"geometry" $\boxed{+G}$

the moduli space $\mathcal{A}_{G,\Sigma}$ of decorated twisted G-local systems on Σ

"algebra"

function ring $\mathcal{O}(\mathcal{A}_{G,\Sigma})$

trace functions along γ

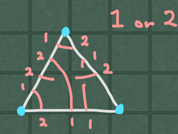
"skein algebra" $\mathcal{S}_{\mathfrak{g},\Sigma}$

c.f. $\mathfrak{g} = \mathfrak{sl}_2$, $\quad = A \,)(\, + A^{-1}$

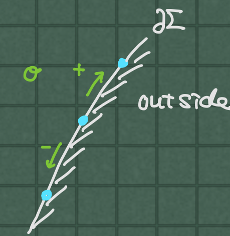
$\bigcirc = (A^2 + A^{-2})\,\emptyset$

§ 1 Stated skein algebras

- Bonahon - Wong introduced a skein with states to define the quantum trace map



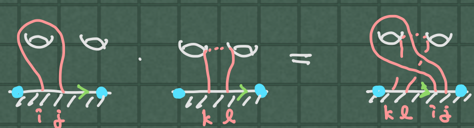
- Lê defined the stated skein algebra $\mathcal{S}_{sl_2,\Sigma}(\mathbb{B}, o)$ Σ ⌟



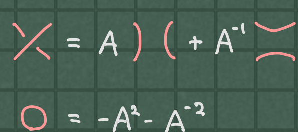
$M \subset \partial\Sigma$: a set of marked points
$\mathbb{B}$: a set of connected components of $\partial\Sigma \setminus M$ (boundary intervals)
$o \in \{+,-\}^{\mathbb{B}}$: a choice of orientation of $\mathbb{B}$
$\Lambda = \{1, 2\}$: the set of states for sl_2

Definition

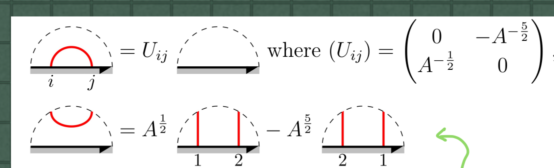
$$\mathcal{S}_{sl_2,\Sigma}(\mathbb{B}, o) := \mathbb{Z}_A\{\text{tangles with states} \in \Lambda\} / \text{skein relation}$$



- Internal skein relation



- Stated skein relation



" the sticking trick "

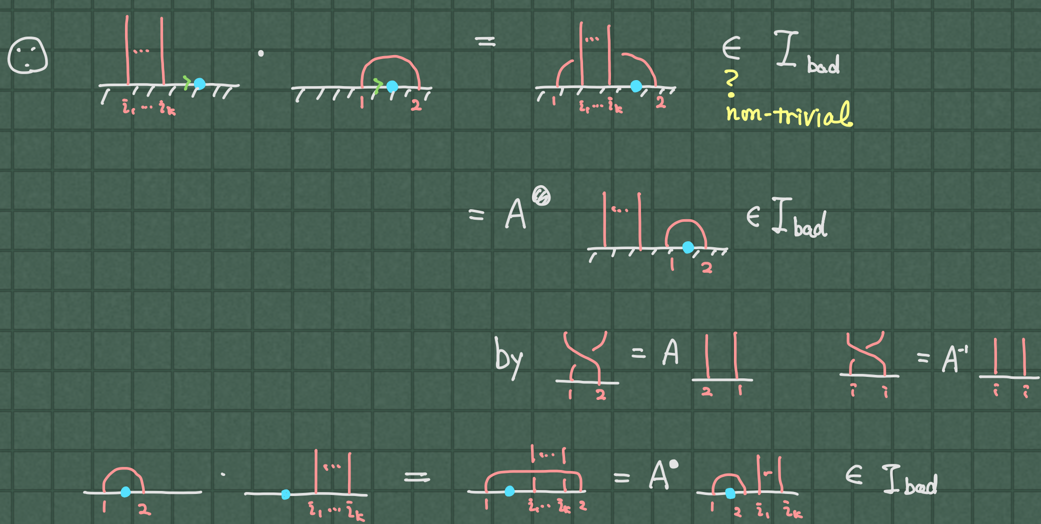
For simplicity, we fix an orientation of $\mathbb{B}$ as positive $o^+ : \mathbb{B} \to \{+\}$

$\mathcal{S}_{sl_2,\Sigma}(\mathbb{B}) := \mathcal{S}_{sl_2,\Sigma}(\mathbb{B}, o^+)$

Let us define a $\mathbb{Z}_A$- submodule $I_{bad} := \mathbb{Z}_A \left\{ \begin{array}{c} \text{stated tangles containing} \\ \text{bad arcs} \end{array} \right\}$

Lemma [Costantino-Lê '19]

$\mathcal{I}_{bad}$ is equal to the left and right ideal generated by bad arcs



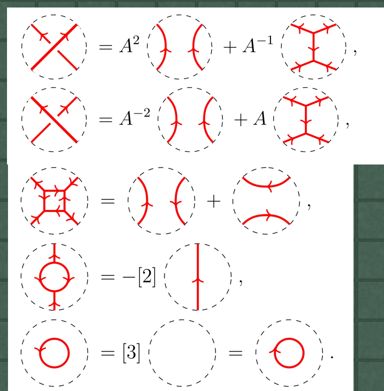
Definition

$\mathcal{S}_{sl_2,\Sigma}(\mathbb{B})_{rd} = \mathcal{S}_{sl_2,\Sigma}(\mathbb{B}) / \mathcal{I}_{bad}$: the reduced stated skein algebra.

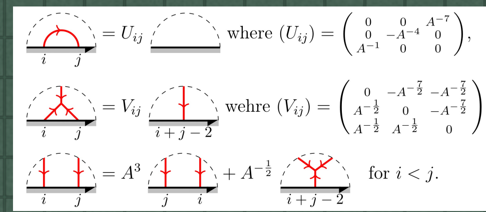
Similarly, we define $\mathcal{S}_{\mathfrak{g},\Sigma}(\mathbb{B})$ and $\mathcal{S}_{\mathfrak{g},\Sigma}(\mathbb{B})_{rd}$ for $\mathfrak{g} = sl_3, sp_4$

◎ $\mathfrak{sl}_3$ (Higgins '20)

$\mathfrak{sl}_3$-web : tangled trivalent graph on Σ with 



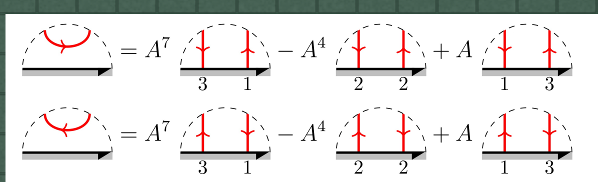
Kuperberg's internal skein relations



the stated skein relations with states $\Lambda = \{1, 2, 3\}$

$\rightsquigarrow \mathcal{S}_{\mathfrak{sl}_3, \Sigma}(\mathbb{B})$

Lemma (the sticking trick)



Lemma (Ishibashi - Y.)

$$I_{bad} := \mathbb{Z}_A \left\{ \begin{array}{l} \mathfrak{sl}_3\text{-webs containing} \\ \text{bad arcs} \end{array} \right.$$

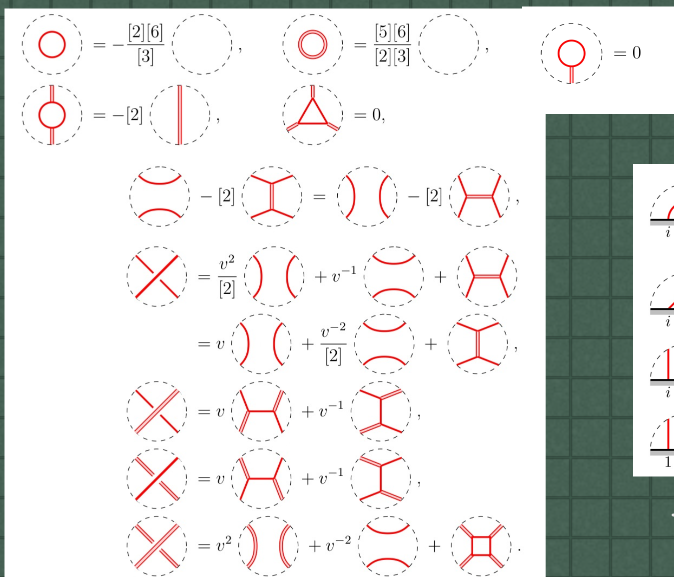
$$\left| \begin{array}{l} i < j \\ i, j \in \Lambda = \{1, 2, 3\} \end{array} \right\}$$

coincides with the left and right ideal generated by bad arcs.

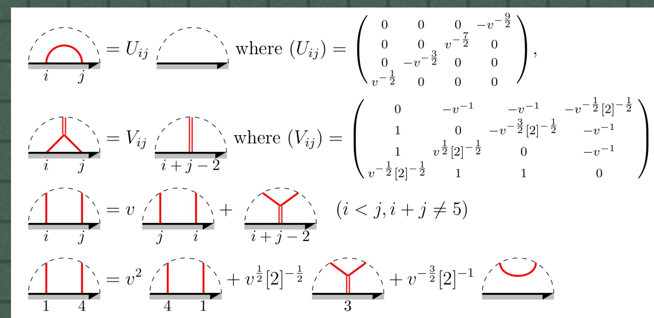
$\longrightarrow \mathcal{S}_{\mathfrak{sl}_3, \Sigma}(\mathbb{B})_{rd}$

◎ sp_4 (Ishibashi - Y.)

sp_4-web : tangled triv. graphs on Σ with ⅄



Kuperberg's internal skein relations

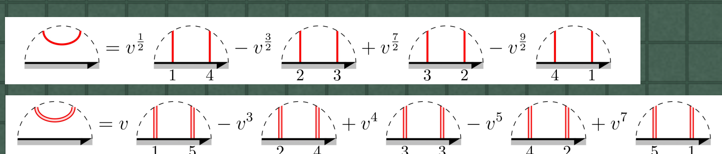


the stated skein relation with states $\Lambda_1 = \{1,2,3,4\}$ for $|$

$\Lambda_2 = \{1,2,3,4,5\}$ for $\|$

$\rightsquigarrow \mathcal{S}_{sp_4,\Sigma}(\mathbb{B})$ $\quad (R_v = \mathbb{Z}[v^{\pm\frac{1}{2}}, 1/[2]]\text{-algebra})$

<u>Lemma</u> (the sticking trick)



<u>Lemma</u> $I_{bad} = R_v \left\{ sp_4\text{-webs containing bad arcs } \middle| \begin{array}{l} i_1 < i_2 \ (i_1, i_2 \in \Lambda_1) \\ j_1 < j_2 \ (j_1, j_2 \in \Lambda_2) \end{array} \right\}$


$i_1 < i_2$ $\quad$ $j_1 < j_2$

coincides with the left and right ideal generated by bad arcs.

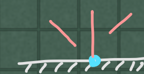
$\rightsquigarrow \mathcal{S}_{sp_4,\Sigma}(\mathbb{B})_{rd}$

§3. the clasped skein algebra $\mathscr{S}_{\mathfrak{g},\Sigma}(M)$

- the clasped skein algebra $\mathscr{S}_{\mathfrak{sl}_2,\Sigma}(M)$ is introduced by Muller ('16)

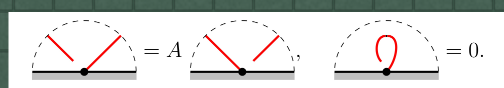
  and he showed $\mathscr{S}_{\mathfrak{sl}_2,\Sigma}(M) = \underline{\mathcal{A}^{q}_{\mathfrak{sl}_2,\Sigma}}$ the quantum cluster algebra

  (quantization of $\mathcal{O}(\mathcal{A}_{\mathfrak{g},\Sigma})$)

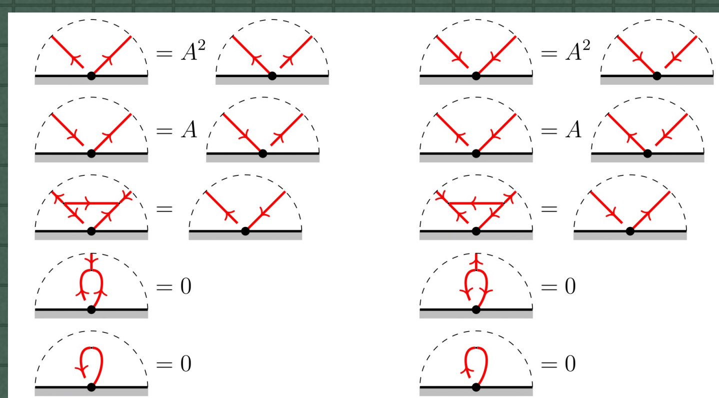
  - endpoints have elevation : 

- the clasped skein relations

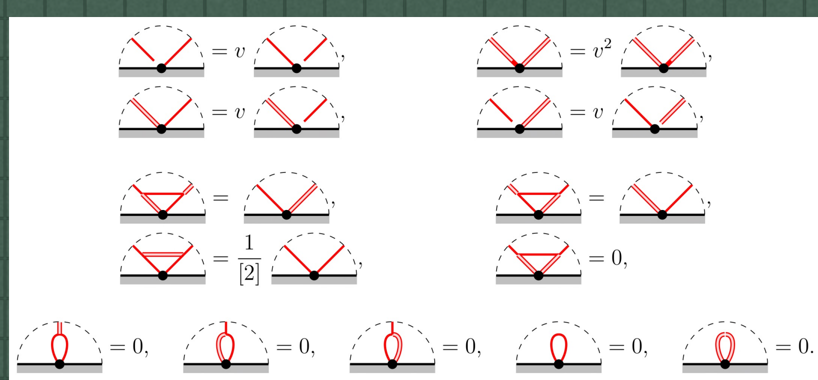
  - $\mathfrak{sl}_2$ (Muller '16)



  - $\mathfrak{sl}_3$ (Frohman-Sikora '20)



  - $\mathfrak{sp}_4$ (Ishibashi-Y '22)



We consider the <u>boundary-localized</u> clasped skein algebra $\mathscr{S}_{\mathfrak{g},\Sigma}(M)[\partial^{-1}]$

which has inverses of boundary web  $:= \Big($  $\Big)^{-1}$

§4. the state-clasp correspondence

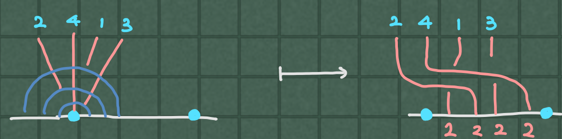
We define algebra homomorphisms

$$\begin{cases} \mathcal{S}_{\mathbb{M}\to\mathbb{B}} : \mathcal{S}_{\mathfrak{g},\Sigma}(\mathbb{M})[\partial^{-1}] \longrightarrow \mathcal{S}_{\mathfrak{g},\Sigma}(\mathbb{B})_{\mathrm{rd}} \\ \mathcal{S}_{\mathbb{B}\to\mathbb{M}} : \mathcal{S}_{\mathfrak{g},\Sigma}(\mathbb{B})_{\mathrm{rd}} \longrightarrow \mathcal{S}_{\mathfrak{g},\Sigma}(\mathbb{M})[\partial^{-1}] \end{cases}$$

s.t. $\mathcal{S}_{\mathbb{B}\to\mathbb{M}} \circ \mathcal{S}_{\mathbb{M}\to\mathbb{B}} = \mathrm{id}_{\mathcal{S}_{\mathfrak{g},\Sigma}(\mathbb{M})[\partial^{-1}]}$, $\mathcal{S}_{\mathbb{M}\to\mathbb{B}} \circ \mathcal{S}_{\mathbb{B}\to\mathbb{M}} = \mathrm{id}_{\mathcal{S}_{\mathfrak{g},\Sigma}(\mathbb{B})_{\mathrm{rd}}}$

◎ $\mathfrak{sl}_2$

$\mathcal{S}_{\mathbb{M}\to\mathbb{B}}$:



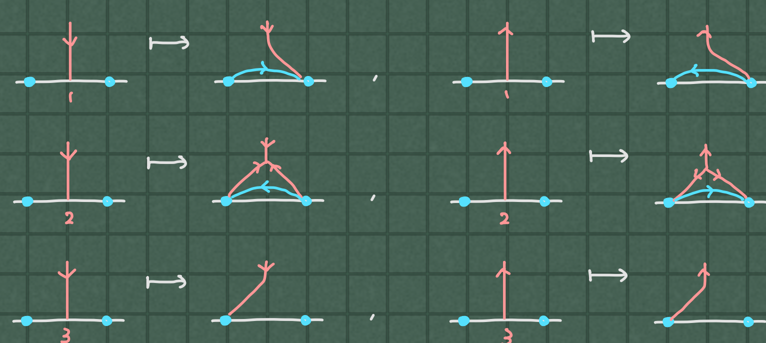
← maximum of states

$\mathcal{S}_{\mathbb{B}\to\mathbb{M}}$:

◎ $\mathfrak{sl}_3$

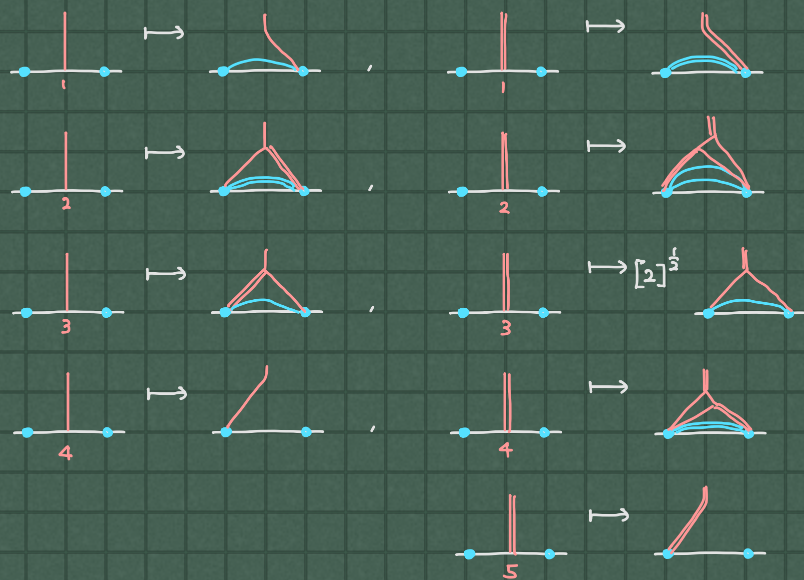
$\mathcal{S}_{\mathbb{M}\to\mathbb{B}}$:

$\mathcal{S}_{\mathbb{B}\to\mathbb{M}}$:



◎ sp_4 $\quad S_{M\to B}$:

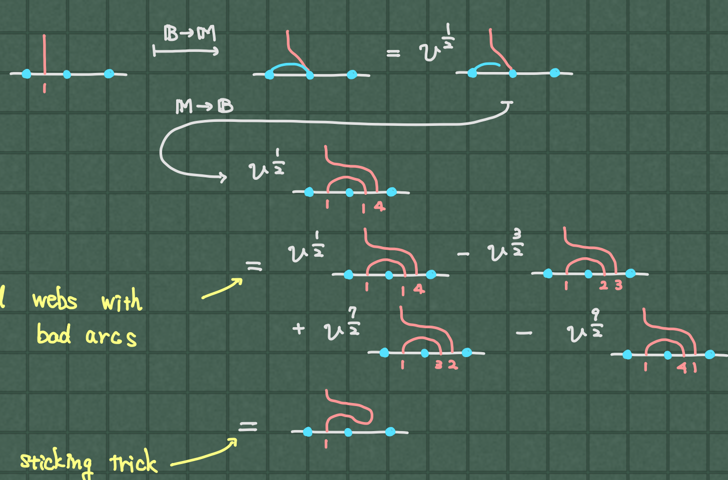
$S_{B\to M}$:



By definition, $\quad S_{B\to M} \circ S_{M\to B} = \mathrm{id}$ is easy.

$S_{M\to B} \circ S_{B\to M} = \mathrm{id}$: use the sticking trick

e.g. sp_4



add webs with bad arcs

the sticking trick