Higher-rank skein algebras and quantum cluster algebras
（高階スケイン代数と量子クラスター代数）

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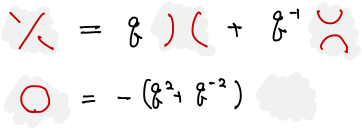
§1 Introduction

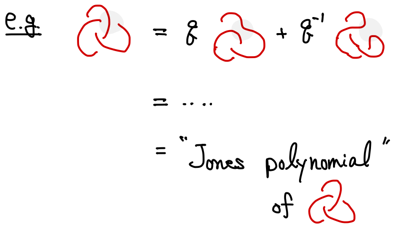
- My research interest "quantum topology"

  - quantum invariants : $J_{\mathfrak{g},V} :$ knots $\longrightarrow \mathbb{Z}_q := \mathbb{Z}[q^{\pm 1}]$
    ↖ rep of $\mathfrak{g}$

e.g. $\mathfrak{g} = \mathfrak{sl}_2$, $V = V_1 = \mathbb{C}_q^2$: 2-dim irrep. of $U_q(\mathfrak{sl}_2)$ $\rightsquigarrow$ Jones polynomial
(V_n : $(n+1)$-dim. irrep) (colored Jones polynomial)

the Kauffman bracket skein relation





① tail of K : $\lim\limits_{n\to\infty} J_{\mathfrak{g},K}(n) =$ " q-series " $\in \mathbb{Z}[[q]]$
$\qquad\qquad \| $
$\qquad J_{\mathfrak{g},V_n}(K)$

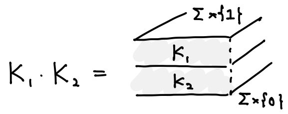
e.g. $K = T(2,m)$ $\rightsquigarrow$ $\lim\limits_{n\to\infty} J_{\mathfrak{g},T(2,m)}(n) =$ "(false) theta series"

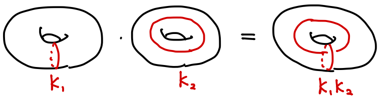
$\rightsquigarrow$ higher-rank $\mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sp}_4, \mathfrak{g}_2$ (rank 2)

$\lim\limits_{n\to\infty} J_{\mathfrak{g},K}(V_{n\lambda}) =$ " higher version of (false) theta series " ↔ quantum modular form ↔ character of VOA

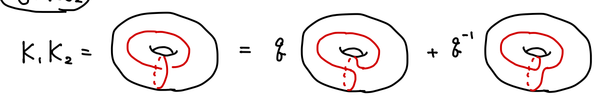
② Skein algebra (Σ : a surface)

$\mathbb{Z}_q\{\text{knots in } \Sigma \times [0,1]\} \xrightarrow{\quad} \mathcal{S}^q_{\mathfrak{g},\Sigma}$
↖ skein relation





$\boxed{\mathfrak{g}=\mathfrak{sl}_2}$



Σ : annulus


$\mathcal{S}^q_{\mathfrak{sl}_2,\Sigma} = \mathbb{Z}_q\{\ \}\cong \mathbb{Z}_q[X]$

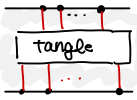
$\rightsquigarrow$ Σ : a marked surface : $\mathcal{S}^q_{\mathfrak{g},\Sigma} \longleftrightarrow \mathcal{A}^q_{\mathfrak{g},\Sigma}$: quantum cluster algebra

§2 skein relations

▨ Web category

$\boxed{\mathfrak{g} = \mathfrak{sl}_2}$: the Kauffman bracket skein relation

$\mathrm{Web}_{\mathfrak{sl}_2}$ $\left\{ \begin{array}{l} \text{Obj : finite points on the interval} \\ \text{Mor : } \mathbb{C}(q)\text{-span of} \end{array} \right.$



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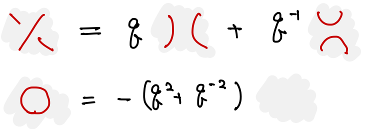
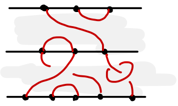


- web category
- Temperley-Lieb category
- spider category

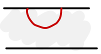
$\mathfrak{sl}_2$-web

Composition



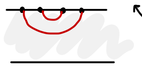
$\mathrm{Web}_{\mathfrak{sl}_2}(\emptyset, n\text{-points}) \cong \mathrm{Inv}_{U_q(\mathfrak{sl}_2)}(\underbrace{V_1 \otimes \cdots \otimes V_1}_{n}) \cong \mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(\mathbb{C}_q, V_1 \otimes \cdots \otimes V_1)$

e.g.



$: \mathbb{C}(q) \longrightarrow V_1 = \mathbb{C}v_1 \oplus \mathbb{C}v_{-1}$

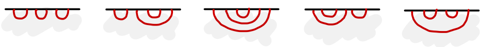
$\quad 1 \longmapsto v_1 \otimes v_{-1} - q^{-1} v_{-1} \otimes v_1$



∃ diagrammatic basis ↔ { non-crossing matchings } $=: \mathrm{BWeb}_{\mathfrak{sl}_2}(n)$

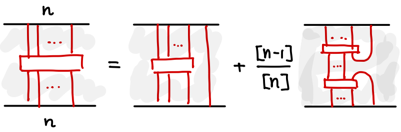
e.g. $n = 6$

$\mathrm{BWeb}_{\mathfrak{sl}_2}(6) = \{$



$\}$

- Jones-Wenzl projectors ("internal clasp")

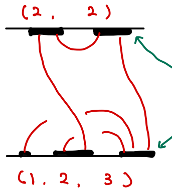


$: \quad V_1^{\otimes n} \xrightarrow{\pi_n} V_n \overset{\iota_n}{\hookrightarrow} V_1^{\otimes n}$

q-symm. power in $V^{\otimes n}$

▨ clasped web category ($\mathfrak{g} = \mathfrak{sl}_2$)

$\widetilde{\mathrm{Web}}_{\mathfrak{sl}_2}$ $\left\{ \begin{array}{l} \text{Obj : } (n_1, n_2, \ldots, n_k) \quad (\mu \vDash n) \\ \text{Mor : morphisms of } \mathrm{Web}_{\mathfrak{sl}_2} + \text{skein relation at "external clasp"} \end{array} \right.$



external clasp

$V_1^{\otimes n} \xrightarrow[\pi_n]{} V_n$

$= 0$

[Kuperberg '96] $\quad \widetilde{\mathrm{Web}}(\emptyset, (n_1, \ldots, n_k)) \cong \mathrm{Inv}_{U_q(\mathfrak{sl}_2)}(V_{n_1} \otimes V_{n_2} \otimes \cdots \otimes V_{n_k}) = \mathrm{Hom}(\mathbb{C}_q, V_{n_1} \otimes \cdots \otimes V_{n_k})$

$\cup$

∃ basis : $\widetilde{\mathrm{BWeb}}(n_1, \ldots, n_k) = \{$non-crossing matchings$\} / \{$

$\}$

▨ rank 2 case [Kuperberg '96]

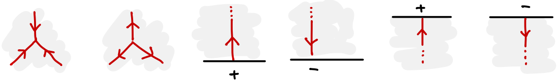
($\mathfrak{sl}_3$)

- $\mathrm{Web}_{\mathfrak{sl}_3}$ {

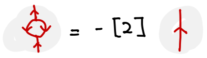
Objects : sequence of $\{+, -\}$

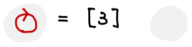
upward ↑ +, downward ↙ −

$\mathfrak{sl}_3$-web = oriented uni-trivalent graphs with crossings

s.t. 

skein relation : 





[Kuperberg '96] $\mathrm{Web}_{\mathfrak{sl}_3}(\emptyset, \varepsilon_1\varepsilon_2\cdots\varepsilon_n) \cong \mathrm{Inv}_{\mathfrak{sl}_3}(V_{\varepsilon_1}\otimes\cdots\otimes V_{\varepsilon_n})$ $(V_+ = V_{\varpi_1},\ V_- = V_{\varpi_2} \cong V_{\varpi_1}^*)$

$\cup$

$\mathrm{BWeb}_{\mathfrak{sl}_3}(\varepsilon_1\cdots\varepsilon_n) = \{$ non-elliptic $\mathfrak{sl}_3$-webs $\}$

↳ elliptic faces: 

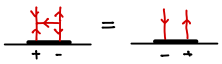
- $\widetilde{\mathrm{Web}}_{\mathfrak{sl}_3}$ {

Obj : $(S_1, S_2, \dots, S_k)$ S_i : a sequence of $\{+, -\}$

Mor : $\mathfrak{sl}_3$-web + skein relation at "external clasps"

[Frohman-Sikora '21, Ishibashi-Y. '22]

S_i : a multiset on $\{+, -\}$

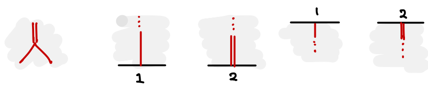


$\widetilde{\mathrm{Web}}_{\mathfrak{sl}_3}(\emptyset, (S_1, S_2, \dots, S_k)) \cong \mathrm{Inv}_{U_q(\mathfrak{sl}_3)}(V_{S_1}\otimes\cdots\otimes V_{S_k})$ $V_S := V_{a\varpi_1 + b\varpi_2}$

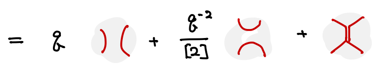
$\left(\begin{array}{l} a := \#\text{ of } + \text{ in } S \\ b := \#\text{ of } - \text{ in } S \end{array}\right)$

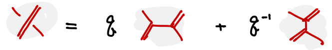
($\mathfrak{sp}_4$) $\mathrm{Web}_{\mathfrak{sp}_4}$ {

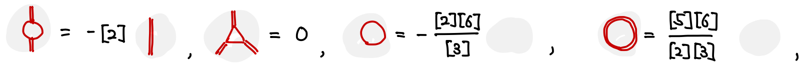
Obj : a sequence of $\{1, 2\}$

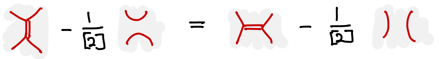
$\mathfrak{sp}_4$-web : 

skein relation : 









[Kuperberg '96] $\mathrm{Web}_{\mathfrak{sp}_4}(\emptyset, \varepsilon_1\varepsilon_2\cdots\varepsilon_n) = \mathrm{Inv}_{U_q(\mathfrak{sp}_4)}(V_{\varepsilon_1}\otimes\cdots\otimes V_{\varepsilon_n})$ $\left(\begin{array}{l} V_1 = V_{\varpi_1} : \text{4-dim irrep} \\ V_2 = V_{\varpi_2} : \text{5-dim irrep.} \end{array}\right)$

$\cup$

$\mathrm{BWeb}_{\mathfrak{sp}_4}(\varepsilon_1\varepsilon_2\cdots\varepsilon_n) = \{$ non-elliptic "crossroad" webs $\}$

↱ replace all  with

elliptic faces :  etc 

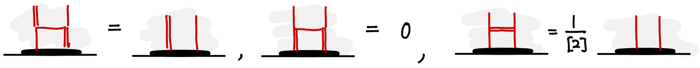
↖ crossroad

(c.f. [Bodish '22] Light Ladder basis

$\mathrm{Web}_{\mathfrak{sp}_4}^{\mathbb{Z}_q[1/[2]]} \longrightarrow U_{\mathbb{Z}_q}(\mathfrak{sp}_4)\text{-mod}$ (c.f. Elias $\mathfrak{gl}_n$))

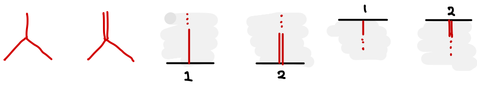
monoidal equivalence

- skein relation at "external clasps" [Ishibashi - Y. '22+]



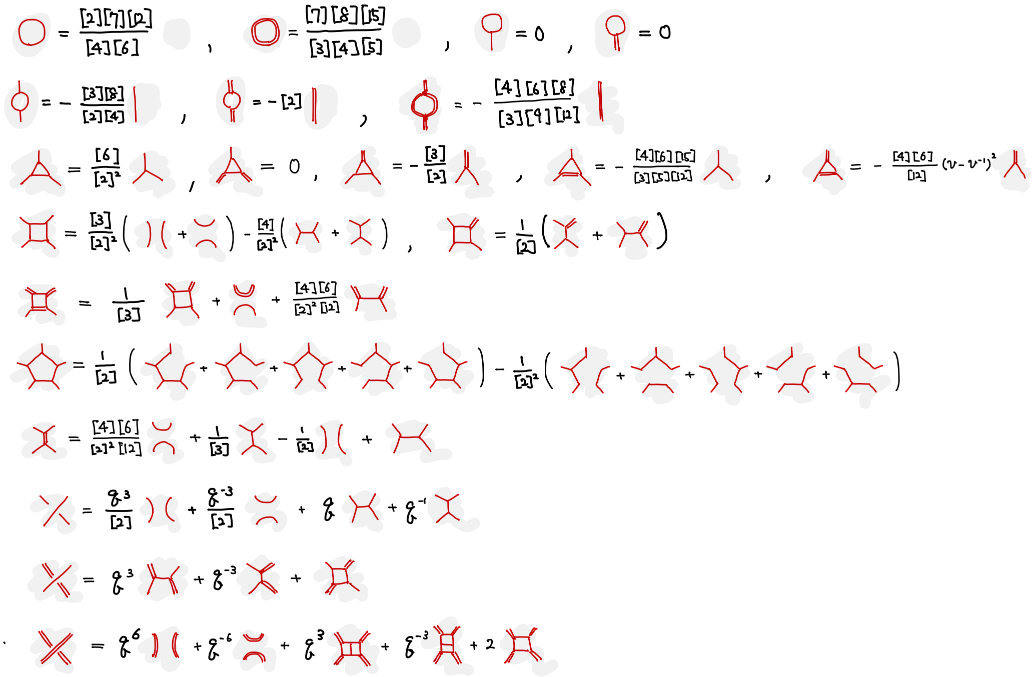
($\mathfrak{g}_2$) $\mathrm{Web}_{\mathfrak{g}_2}$ { Obj : a sequence of $\{1, 2\}$

$\mathfrak{g}_2$- web



skein rel.

[Sakamoto -Yonezawa '17]



[Kuperberg '96 (?)] $\mathrm{Web}_{\mathfrak{g}_2}(\emptyset, \varepsilon_1\varepsilon_2 \cdots \varepsilon_n) = \mathrm{Inv}_{U_q(\mathfrak{g}_2)}(V_{\varepsilon_1} \otimes \cdots \otimes V_{\varepsilon_n})$ $\left(\begin{array}{l} V_1 = V_{\varpi_1} : 7\text{-dim irrep} \\ V_2 = V_{\varpi_2} : 14\text{-dim irrep.} \end{array} \right)$

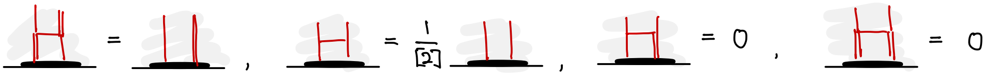
$\cup$

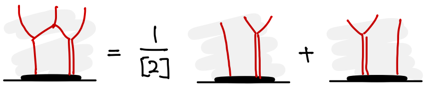
$\mathrm{BWeb}_{sp_4}(\varepsilon_1\varepsilon_2 \cdots \varepsilon_n) = \{$ non-elliptic no internal ‖ webs $\}$

$\uparrow$

[Sikora-Westbury '07]

- skein relation at "external clasps"

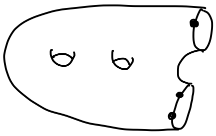




$\leftarrow$ $\dim(\mathrm{Hom}(V_{\varpi_1}^{\otimes 3}, V_{\varpi_1+\varpi_2})) = 2$

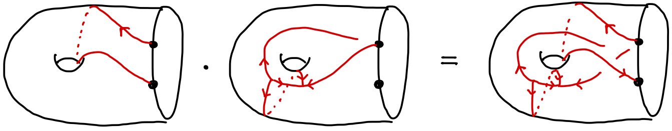
§ Skein & Cluster algebra.

▨ skein algebra of marked surface $\Sigma =$

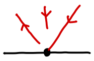


$$\mathcal{S}^{q}_{\mathfrak{g},\Sigma} = \mathcal{R}\{\mathfrak{g}\text{-webs on } \Sigma\} \Big/ \begin{array}{l}\text{skein relation in } \Sigma \setminus \partial\Sigma \\ \text{“skein relation at marked points”}\end{array}$$

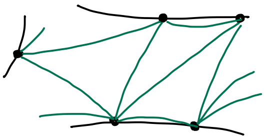
- multiplication



- “marked points” ⟷ “external clasps” ⇝ • external clasps define skein relation at marked points



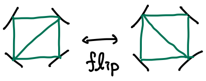
▨ (quantum) cluster algebra of Σ



$\in \mathrm{Tri}(\Sigma)$

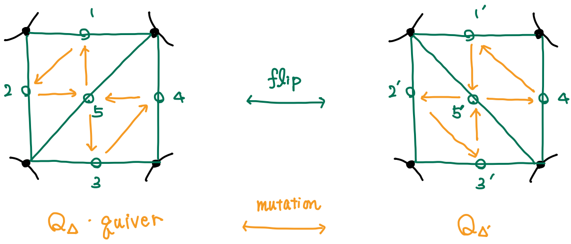
$\mathrm{Tri}(\Sigma)$: the set of ideal triangulation of Σ

Remark $\forall \Delta, \Delta' \in \mathrm{Tri}(\Sigma)$ Δ is related to Δ' by flips



quantum torus

$$\mathcal{A}^{q}_{\mathfrak{sl}_3,\Sigma} := \bigsqcup_{\Delta\in \mathrm{Tri}(\Sigma)} \mathbb{T}_\Delta \Big/ \text{“exchange relation”}$$: the quantum cluster algebra of Σ

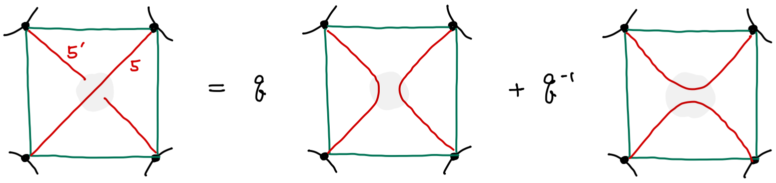


$i \in$ boundary arc A_i : invertible (frozen variable)

$\mathbb{T}_\Delta := \langle \{A_i \mid i \in Q_\Delta\}^{\pm} \rangle$ cluster

$$\begin{cases} A_5 A_{5'} = A_2 A_4 + A_1 A_3 \\ A_i = A_{i'} \quad (i \neq 5) \end{cases}$$: exchange relation

in $\underline{\mathcal{S}^{q}_{\mathfrak{sl}_3,\Sigma}}$



[Muller 2016]

- $\mathcal{A}^{q}_{\mathfrak{sl}_3,\Sigma} \subset \mathcal{S}^{q}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \subset \mathcal{U}^{q}_{\mathfrak{sl}_3,\Sigma} \subset \mathrm{Frac}\, \mathcal{S}^{q}_{\mathfrak{sl}_3,\Sigma}$

localization at ∂-arcs

upper cluster algebra

$\mathcal{U}_{\mathfrak{sl}_3,\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{\mathfrak{sl}_3,\Sigma})$

- $\mathcal{A}^{q} = \mathcal{U}^{q}$ (acyclic exchange type)

⇝ $\mathcal{A}^{q}_{\mathfrak{sl}_3,\Sigma} = \mathcal{S}^{q}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] = \mathcal{U}^{q}_{\mathfrak{sl}_3,\Sigma}$

▨ Main results

[Ishibashi - Y. 2023] $\mathscr{S}^{q}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \subset \mathscr{A}^{q}_{\mathfrak{sl}_3,\Sigma} \subset \mathrm{Frac}\,\mathscr{S}^{q}_{\mathfrak{sl}_3,\Sigma}$

(c.f. [Fomin - Pylyavskyy '16])

[I - Y. 2022 +] $\mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}] \subset \mathscr{A}^{q}_{\mathfrak{sp}_4,\Sigma} \subset \mathrm{Frac}\,\mathscr{S}^{q}_{\mathfrak{sp}_4,\Sigma}$

$\cap$ $\mathbb{Z}_q$-subalgebra

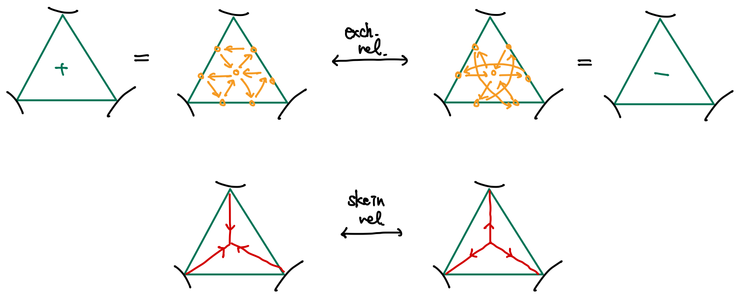
$\mathscr{S}^{q}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}]$

[I-Y. in progress] $\mathscr{A}^{q}_{\mathfrak{g}_2,\Sigma} \subset \mathrm{Frac}\,\mathscr{S}^{q}_{\mathfrak{g}_2,\Sigma}$

▨ Difficulty for higher-rank case

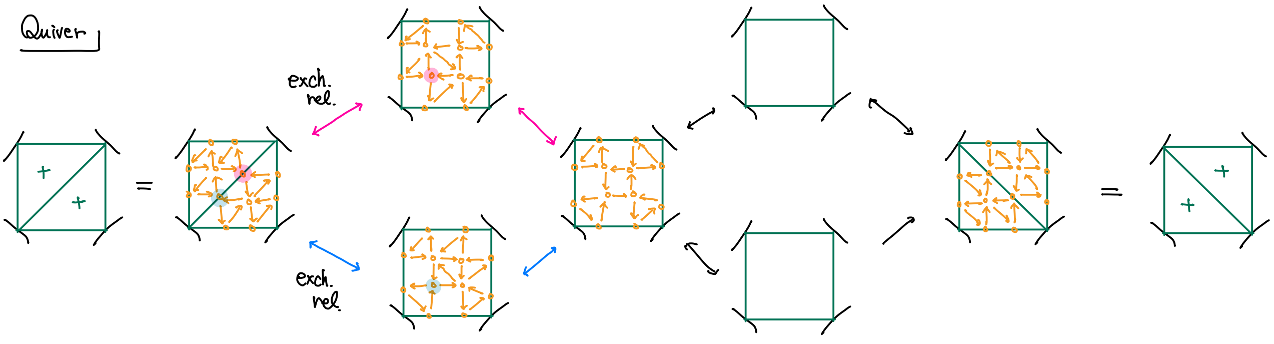
$\mathrm{Tri}_{\mathfrak{sl}_3}(\Sigma) \ni \Delta$: decorated ideal triangulation

- switch of decoration

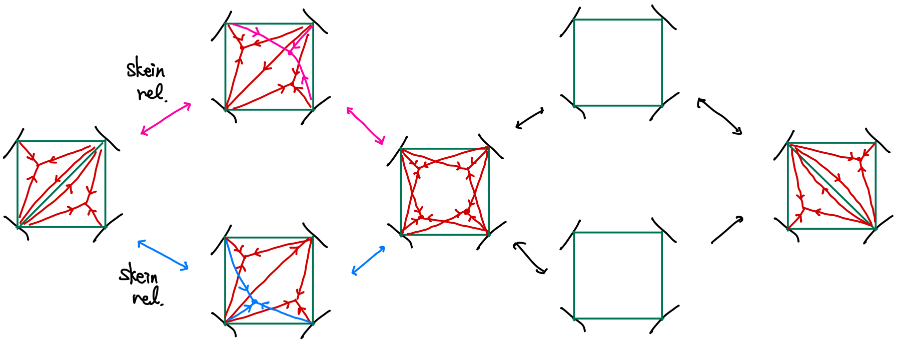


- Construction of flip as $\mathfrak{sl}_3$-webs.

Quiver



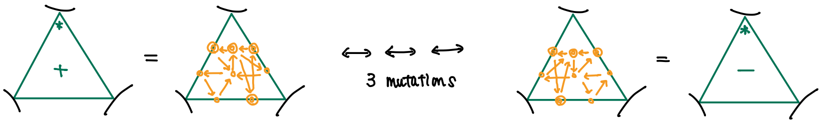
$\mathfrak{sl}_3$-web



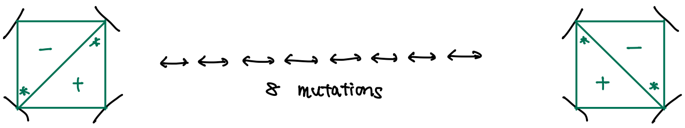
- Expand any $\mathfrak{sl}_3$-webs as a polynomial of known cluster variables

$\mathfrak{sp}_4$-case

- decoration



- flip.



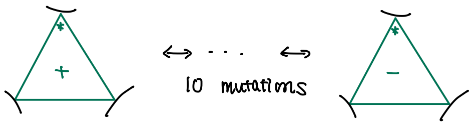
☆ $\mathcal{S}^{q}_{\mathfrak{sp}_4,\Sigma}$: $\mathbb{Z}_q[1/[2]]$ - algebra

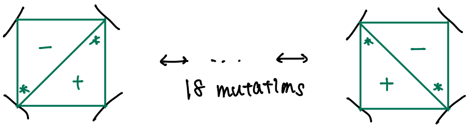
$\mathcal{A}^{q}_{\mathfrak{sp}_4,\Sigma}$: $\mathbb{Z}_q$ - algebra

related to Lusztig's integer form ?

$\rightsquigarrow$ Define $\mathbb{Z}_q$ - subalgebra $\mathcal{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}$ of $\mathcal{S}^{q}_{\mathfrak{sp}_4,\Sigma}$

$\mathfrak{g}_2$ -case





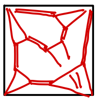
§ Conjectures, other works,

<u>Conjecture.</u>

- $\mathcal{A}^{q}_{\mathfrak{g},\Sigma} = \mathcal{S}^{q}_{\mathfrak{g},\Sigma}[\partial^{-1}]$ for $\mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sp}_4, \mathfrak{g}_2$

  (※. $q = 1$ is OK for $\mathfrak{sl}_3, \mathfrak{sp}_4$
  via $\mathcal{A} = \mathcal{U}$ theorem in [Ishibashi - Oya - Shen '23])

- {cluster variables} = { tree-type webs }

  e.g. $\mathfrak{sp}_4$  is a cluster variable

<u>Problem.</u>

- Construct positive basis (c.f. [Mandel - Qin '23+] $\mathfrak{g} = \mathfrak{sl}_2$
  theta basis = bracelet basis)

<u>Other work</u> [Ishibashi - Kano - Y. 24+] $\mathfrak{g} = \mathfrak{sl}_2$

the skein algebra of a "walled surface"

≅ the quantum cluster algebra with coefficients