The tail of the one-row colored $\mathfrak{sl}(3)$ Jones polynomial and the Andrews-Gordon type identity

July, 09, 2020 表現論セミナー

RIMS, Kyoto Univ.
Wataru YUASA

§1 Quick Introduction to Quantum Invariants of Knots and Links

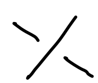
Diagrammatic definition of knots & links

an ℓ-component link diagram D

$\underset{def}{\Leftrightarrow}$ $D : \coprod^{\ell} S^1 \longrightarrow \mathbb{R}^2$, an generic immersion

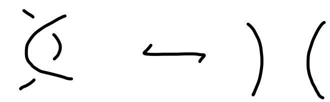
with over/under information on intersection points.

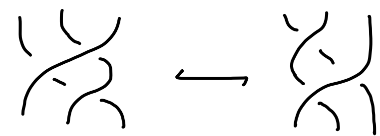
i.e intersection points

= 

Reidemeister moves

(Ω1) 

(Ω2) 

(Ω3) 

the set of ℓ-component links

$:= \{ \ell\text{-component link diagrams} \} \Big/ \begin{matrix} (\Omega 1) \\ (\Omega 2) \\ (\Omega 3) \\ + \text{isotopy} \end{matrix}$

i.e. $[D] = [D']$

D and D′ are related by
a finite sequence of (Ω1), (Ω2) (Ω3)
+ isotopy

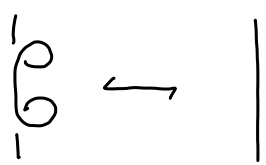
Oriented links

- link diagram + orientation on the image of S^1
- equivalence : orientation preserving $(\Omega 1)$ $(\Omega 2)$ $(\Omega 3)$ moves

framed links

- equivalence : $(\Omega 0)$ $(\Omega 2)$ $(\Omega 3)$ moves

$(\Omega 0)$



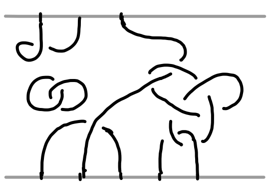
Quantum invariants of knots and links

a **(k, ℓ) - tangle diagram**

$\overset{\Leftrightarrow}{\text{def}}$ a generic immersion of arcs & loops into $\mathbb{R} \times [0,1]$

s.t. • intersection point = ╳

• $\partial\{\text{arcs}\} = \{(1,0),(2,0),\dots,(k,0)\}$
$\cup \{(1,1),(2,1),\dots,(\ell,1)\}$



(5, 3) - tangle diagram

the set of **framed (k, ℓ) - tangles**

$:= \{\, (k,\ell)\text{- tangle diagram} \,\} \big/ (\Omega 0), (\Omega 2), (\Omega 3), \text{isotopy}$

Fix a strict ribbon category $\mathcal{C}$

$$\left[\begin{array}{lll} \text{braiding} & c_{V,W} : V \otimes W \to W \otimes V & \\ \text{duality} & \left\{\begin{array}{l} * : V \longrightarrow V^* \\ b_V : \mathbb{1} \longrightarrow V \otimes V^* \\ d_V : V^* \otimes V \to \mathbb{1} \end{array}\right. & \\ \text{twist} & \theta_V : V \to V, \ \text{isom.} & \end{array}\right]$$

a **category of $\mathcal{C}$ - colored framed tangles $\mathcal{T}_{\mathcal{C}}$**

$\overset{\Leftrightarrow}{\text{def}}$ Obj $\mathcal{T}_{\mathcal{C}}$: a finite sequence of Obj $\mathcal{C} \times \{+,-\}$

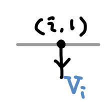
※ $\varnothing$: the empty sequence.

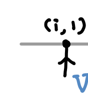
Obj $\mathcal{T}_{\mathcal{C}} \ni \eta_1 = ((V_1, \varepsilon_1), (V_2, \varepsilon_2), \dots, (V_k, \varepsilon_k))$

$\eta_2 = ((W_1, \delta_1), (W_2, \delta_2), \dots, (W_\ell, \delta_\ell))$

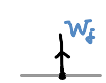
Mor (η_1, η_2) : an oriented framed (k,ℓ) -tangle

s.t. • each component is colored by Obj $\mathcal{C}$

• 

if $\varepsilon_i = +$, 

if $\varepsilon_i = -$

if $\delta_j = +$, 

if $\delta_j = -$

Fix a strict ribbon category $\mathcal{C}$

[braiding $c_{V,W} : V\otimes W \to W\otimes V$

duality $\begin{cases} * : V \longrightarrow V^* \\ b_V : \mathbb{1} \longrightarrow V\otimes V^* \\ d_V : V^*\otimes V \to \mathbb{1} \end{cases}$

twist $\theta_V : V \to V$, isom.]

a category of $\mathcal{C}$-colored framed tangles $\mathcal{T}_\mathcal{C}$

- Obj $\mathcal{T}_\mathcal{C}$: a finite sequence of Obj $\mathcal{C} \times \{+,-\}$

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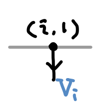
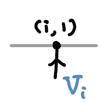
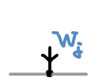
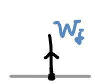
  • 

 if $\varepsilon_i = +$, 

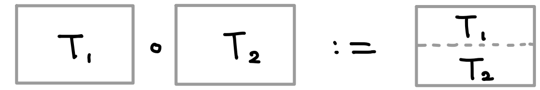
 if $\varepsilon_i = -$

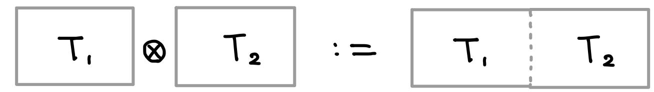
 if $\delta_j = +$, 

 if $\delta_j = -$

- composition



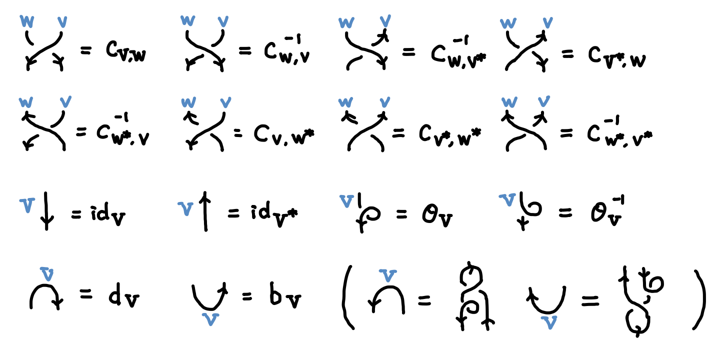
- tensor product



Theorem[Reshetikhin-Turaev, 1990]

$\exists !\ F_\mathcal{C} : \mathcal{T}_\mathcal{C} \longrightarrow \mathcal{C}$, a $\otimes$-preserving functor

s.t. $F((V,+)) = V$, $F((V,-)) = V^*$



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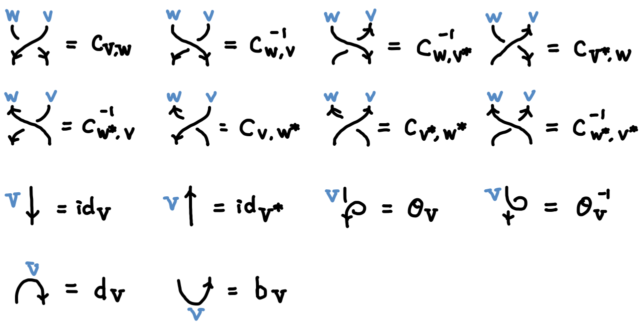


$\rightsquigarrow$ L : an $\mathcal{C}$-colored framed link.

$\Rightarrow F_{\mathcal{C}}(L) \in \mathrm{End}(\mathbb{1})$

e.g. (the quantum $\mathfrak{g}$ invariant of framed links, the colored $\mathfrak{g}$ Jones polynomial)

$\mathfrak{g}$: a simple Lie algebra

$\mathrm{Rep}_f U_q(\mathfrak{g})$: the category of finite dimensional representations of the quantum group $U_q(\mathfrak{g})$. ($q =$ "q^2" : a formal variable)

$\mathrm{Rep}_f U_q(\mathfrak{g}) = \mathcal{C}$

$L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_\ell$: a framed link

$V_i \in \mathrm{Rep}_f U_q(\mathfrak{g})$ is a coloring of L_i

the $(\mathfrak{g}, (V_1, \ldots, V_\ell))$-colored Jones polynomial

$J_L^{\mathfrak{g}}(V_1, V_2, \ldots, V_\ell) \in \mathbb{C}(q^{\frac{1}{2D}})$ is defined by

$$F_{\mathcal{C}}(L)(1) = J_L^{\mathfrak{g}}(V_1, V_2, \ldots, V_\ell) \cdot 1$$

(※ D is determined by $\mathfrak{g}$)

$\square$

§2 Stability of the Colored Jones Polynomial

Properties of the CJP

Theorem [Lê, 2000] (Integrality)

$$J_L^{\mathfrak{g}}(V_{\lambda_1}, V_{\lambda_2}, \dots, V_{\lambda_\ell}) \in q^{\frac{p}{2}}\mathbb{Z}[q^{\pm 1}]$$

Theorem[Garoufalidis-Lê, 2005]

For a knot K,

$\{J_K^{\mathfrak{g}}(\lambda)\}_\lambda$ is q-holonomic (except G_2)

$\{f_n(q) \in \mathbb{Z}((q))\}_n$ is q-holonomic

if $\exists\, d \in \mathbb{N}$, $\exists\, a_\ell \in \mathbb{Z}[u,v]$ $(0 \le \ell \le d)$

s.t. $\displaystyle\sum_{\ell=0}^{d} a_\ell(q, q^n) f_{n+\ell}(q) = 0$

$(\forall n \ge 0)$

e.g. $K =$ 

(n+1) - dim. irrep.

$$J_K^{\mathfrak{sl}_2}(n) = \frac{q^{-\frac{n-1}{2}}}{1-q^{-1}} \sum_{k=0}^{n-1} (1-q^{-n})(1-q^{1-n})\cdots(1-q^{k-n})$$

then,

$$J_K^{\mathfrak{sl}_2}(n-1) = \frac{q^{n-1} + q^{4-4n} - q^{-n} - q^{1-2n}}{q^{\frac{1}{2}}(q^{n-1} - q^{2-n})} J_K^{\mathfrak{sl}_2}(n)$$

$$+ \frac{q^{4-4n} - q^{3-2n}}{q^{2-n} - q^{n-1}} J_K^{\mathfrak{sl}_2}(n+1)$$

- q-holonomic

$\rightsquigarrow$ $\delta_K(n) := \max\deg_q \left(J_K^{\mathfrak{sl}_2}(n) \right)$

$\delta_K^*(n) := \min\deg_q \left(J_K^{\mathfrak{sl}_2}(n) \right)$

are quadratic quasi-polynomials

i.e. $\delta_K^*(n) = a_K^*(n) n^2 + b_K^*(n) n + c_K^*(n)$

s.t. $a_K(n)$, $b_K(n)$, $c_K(n)$

are periodic functions

for $n \gg 0$

$\rightsquigarrow$ (strong) slope conjecture

"$\{a_K(n)\} \cup \{a_K^*(n)\}$"

$\subset \{ \text{slopes of } \partial\Sigma \text{ s.t } \Sigma \subset S^3 \setminus K \}$

"$\{b_K(n)\} \cup \{b_K^*(n)\}$" $\subset \{ \chi(\Sigma) / |\partial\Sigma| \}$

Notation

- For $\underline{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$

$J_{L,\underline{\lambda}}^{\mathfrak{g}}(q) := J_L^{\mathfrak{g}}(V_{\lambda_1}, V_{\lambda_2}, \dots, V_{\lambda_\ell})$

- $\delta_L^*(\underline{\lambda}) := \min\deg J_{L,\underline{\lambda}}^{\mathfrak{g}}(q)$

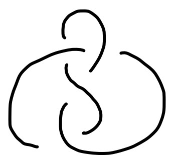
- $\hat{J}_{L,\underline{\lambda}}^{\mathfrak{g}}(q) := \pm q^{-\delta_L^*(\underline{\lambda})} J_{L,\underline{\lambda}}^{\mathfrak{g}}(q) \in \mathbb{Z}[q]$

$= \underbrace{a_0}_{\vee\, 0} + a_1 q + a_2 q^2 + \cdots$

Stability for $\{\hat{J}^{\mathfrak{sl}_2}_{L,n}(q)\}_n$

conjectured by Dasbach - Lin

& proved by $\begin{cases} \text{Armond} \\ \text{Garoufalidis - Lê} \end{cases}$

e.g. $K =$ 

- table of the coefficients of $\hat{J}^{\mathfrak{sl}_2}_{K,n}(q)$ for $n = 2, 3, 4$

	q^0	q^1	q^2	$\cdots$														
$n=2$	1	-1	-1	0	2	0	-2	0	3	0	-3	0	3	0	-3	0	$\cdots$	
$n=3$	1	-1	-1	0	0	3	-1	-1	-1	-1	5	-1	-2	-2	-1	6	$\cdots$	
$n=4$	1	-1	-1	0	0	1	2	0	-2	-1	-1	1	3	1	-2	-3	$\cdots$	

- 0 - stability (the tail of K)

$n=2$	1	-1	-1	0	2	0	-2	0	3	0	-3	0	3	0	-3	0	$\cdots$				
$n=3$	1	-1	-1	0	0	3	-1	-1	-1	-1	5	-1	-2	-2	-1	6	$\cdots$				
$n=4$	1	-1	-1	0	0	1	2	0	-2	-1	-1	1	3	1	-2	-3	$\cdots$				
$\downarrow \lim$																					
Φ_0	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0	-1	$\cdots$				

tail of $\{\hat{J}^{\mathfrak{sl}_2}_{K,n}(q)\}_n$

- 1 - stability

$n=2$	0	0	2	-1	-2	-1	3	0	-3	0	4	0	-3	1	$\cdots$
$n=3$	0	0	2	-1	-2	-1	-1	5	-1	-1	-2	-1	7	$\cdots$	
$n=4$	0	0	2	-1	-2	-1	-1	1	4	1	-2	-2			
Φ_1	0	0	2	-1	-2	-1	-1	1	$\cdots$						

Theorem[Armond, 2013]

L : an A-adequate link.

then, $\exists\, J_L^{\mathfrak{sl}_2}(q) \in \mathbb{Z}[[q]]$ s.t

$$\hat{J}_{L,n}^{\mathfrak{sl}_2}(q) - J_L^{\mathfrak{sl}_2}(q) \in q^{n+1}\mathbb{Z}[[q]]$$

tail of $\{\hat{J}_{K,n}^{\mathfrak{sl}_2}(q)\}$

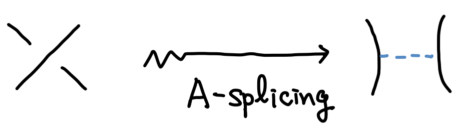
Theorem[Armond-Dasbach, 2016]

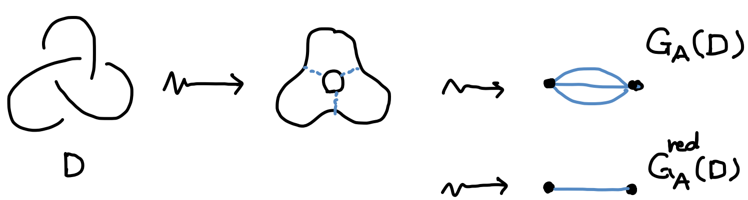
L : an A-adequate link

D : an A-adequate link diagram of L

then, $J_L^{\mathfrak{sl}_2}(q)$ only depends on $G_A^{red}(D)$

e.g





Theorem[Garoufalidis-Lê, 2015]

L : an alternating link.

then $\{\hat{J}_{L,n}^{\mathfrak{sl}_2}(q)\}$ is k-stable ($\forall k \geq 0$)

- $\{f_n(q) \in \mathbb{Z}[[q]]\}_n$ is k-stable

  if $\exists\,\Phi_0(q), \ldots, \exists\,\Phi_k(q) \in \mathbb{Z}((q))$

  s.t. $\displaystyle\lim_{n\to\infty} q^{-k(n+1)}\left(f_n(q) - \sum_{j=0}^{k}\Phi_j(q)\, q^{j(n+1)}\right) = 0$

- $\displaystyle\lim_{n\to\infty} f_n(q) = \Phi(q)$

  $\overset{}{\underset{def}{\Longleftrightarrow}}$ $\forall m \in \mathbb{N}$, $\exists\, N_m \in \mathbb{N}$

  s.t. $f_{N_m}(q) - \Phi(q) \in q^m\mathbb{Z}[[q]]$

Theorem[Garoufalidis-Vuong, 2017]

K : a torus knot

$\mathfrak{g}$: a rank 2 simple Lie algebra.

then, $\{ J^{\mathfrak{g}}_{K,n\lambda}(q) \}_n$ is k-stable ($\forall k$)

$\mathfrak{sl}_3$ $\lambda = (s,t)$ $s,t \in \mathbb{Z}_{\geq 0}$

$\{(ns,nt)\}$

$\{(n,0)\}_n$

Theorem[Y.]

L : a "minus-adequate" oriented link.

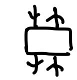
$\exists\, J^{\mathfrak{sl}_3}_L(q) \in \mathbb{Z}[[q]]$ s.t.

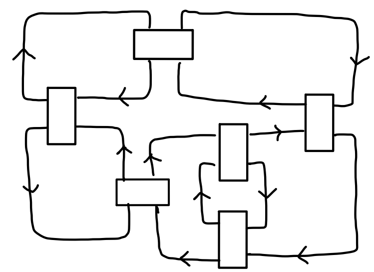
$$\hat{J}^{\mathfrak{sl}_3}_{L,(n,0)}(q) - J^{\mathfrak{sl}_3}_L(q) \in q^{n+1}\mathbb{Z}[[q]]$$

i.e. $\{ J^{\mathfrak{sl}_3}_{L,(n,0)}(q) \}$ is zero stable

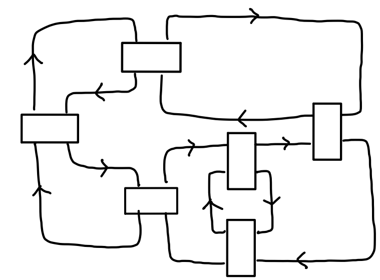
e.g. (minus-adequate)

1. Consider an adequate 4-valent graph with oriented edges

s.t. vertex =  or  or 

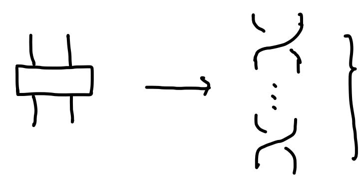


adequate



inadequate

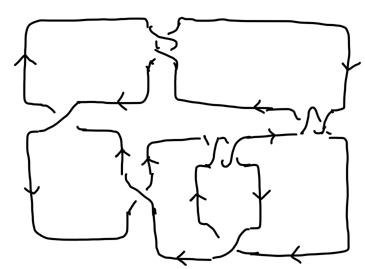
2. Replace



p crossings

($p \in \mathbb{Z}_{\geq 1}$)

e.g.



§ 3 Tails and Andrews-Gordon Type Identities

Graphical Calculus

Problem

Compute the colored Jones polynomial and its tail.

$\mathcal{C} = \mathrm{Rep}_f U_q(\mathfrak{g})$

$\mathcal{T}_{\mathcal{C}} = \mathcal{C}$-colored framed tangles

$$\mathcal{T}_{\mathcal{C}} \xrightarrow{F_{\mathcal{C}}} \mathrm{Rep}_f U_q(\mathfrak{g})$$

$\mathrel{\rotatebox{90}{$\simeq$}}$ **diagrammatic category**

linear skein theory

① Temperley-Lieb category
Web

② Quantum spin network
trivalent graph
+ quantum 6j symbol

"Linear skein theory"

= a functor from $\mathcal{T}_{\mathcal{C}}$ to a diagrammatic representation of $\mathrm{Fun}\,\mathrm{Rep}_f U_q(\mathfrak{g})$

or

$\mathrm{Kar}(\mathrm{Fun}\,\mathrm{Rep}_f U_q(\mathfrak{g}))$

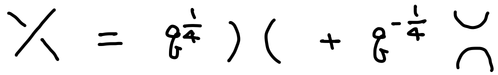
the Kauffman bracket ($\mathfrak{g} = \mathfrak{sl}_2$)

(Notation) $\big| \;=\; \big|^{1} \;=\; \big| \, V_2 \cong V_2^*$

the 2-dim. irreducible rep.

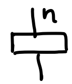
$\big|^{n} = \underbrace{\big|\big|\cdots\big|}_{n} = \big| \, V_2 \otimes \cdots \otimes V_2$

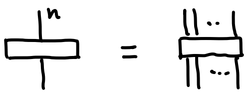
the Kauffman bracket ($\mathfrak{g}=\mathfrak{sl}_2$)



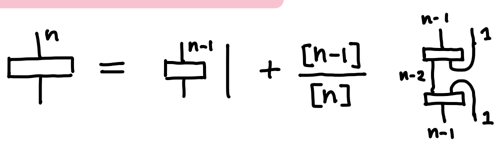
$\bigcirc = -[2]\,\emptyset = (q^{\frac12}+q^{-\frac12})\,\emptyset$

• construction of the color V_{n+1}

= the Jones-Wenzl projector 

i.e  : $V_2^{\otimes n} \twoheadrightarrow \mathrm{Sym}^n V_2 \hookrightarrow V_2^{\otimes n}$

Diagramatic definition



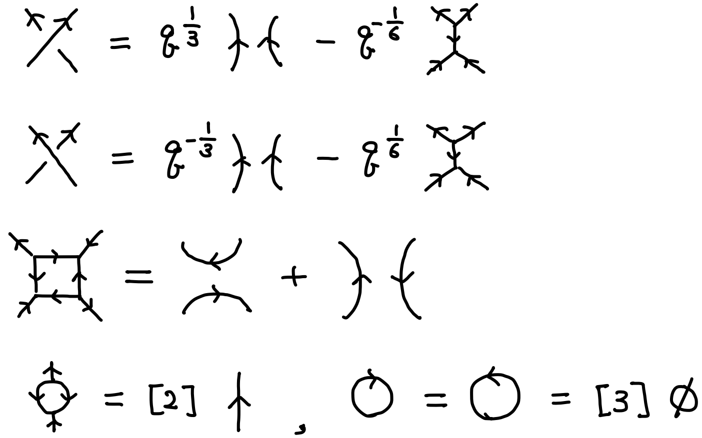
$$\left([n] = \frac{q^{\frac n2}-q^{-\frac n2}}{q^{\frac12}-q^{-\frac12}} \right)$$

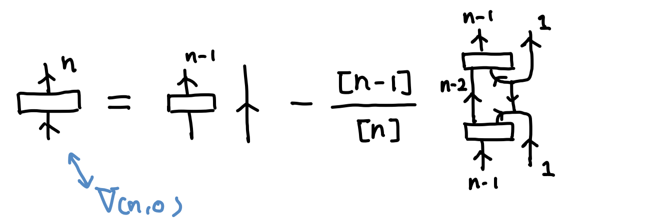
$\rightsquigarrow$ K = 

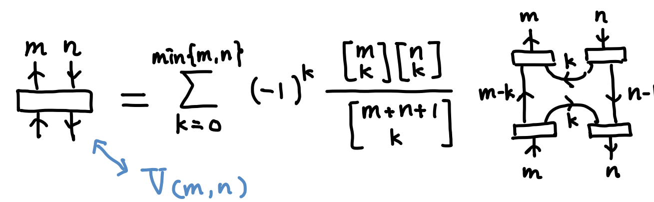
then,  $= J^{\mathfrak{sl}_2}_{K,n}(q)\,\emptyset$

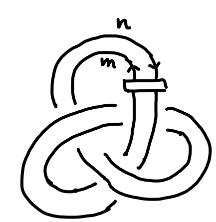
the A_2 bracket ($\mathfrak{g}=\mathfrak{sl}_3$)

(Notation) 



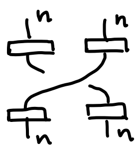




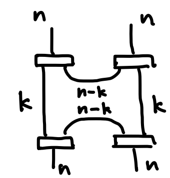
$\rightsquigarrow$  $= J^{\mathfrak{sl}_3}_{K,(m,n)}(q)\,\emptyset$

Twist formulas ($\mathfrak{g} = \mathfrak{sl}_2$)

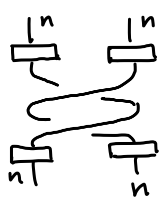
Theorem[Yamada, 1989]



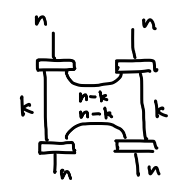
$$= \sum_{k=0}^{n} q^{\frac{1}{4}(-n^2+2k)} \frac{(q)_n}{(q)_k (q)_{n-k}}$$



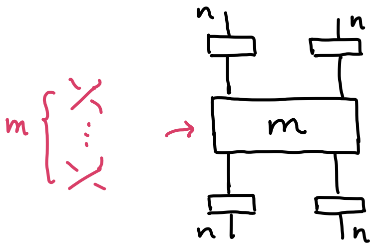
Theorem[Masbaum, 2003]



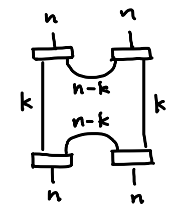
$$= \sum_{k=0}^{n} (-1)^{n-k} q^{\frac{1}{2}(-n^2-n+2k^2+k)} \frac{(q)_n^2}{(q)_k^2 (q)_{n-k}}$$

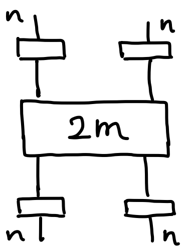


Theorem[Y. 2017]

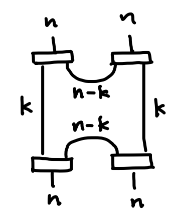


$$= (-1)^{n-k_m} q^{\frac{n-k_m}{2}} \sum_{n \geq k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} (-1)^{\sum_{i=1}^{m} k_i} q^{\frac{1}{2}\sum_{i=1}^{m}(k_i^2+k_i)} \times \frac{(q)_n}{(q)_{n-k_1}(q)_{k_1-k_2}\cdots(q)_{k_{m-1}-k_m}(q)_{k_m}}$$



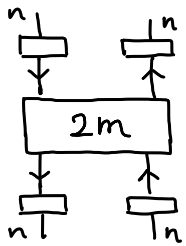


$$= q^{-\frac{m}{2}(n^2+2n)} \sum_{n \geq k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} (-1)^{n-k_m} q^{\frac{n-k_m}{2}} q^{\sum_{i=1}^{m}(k_i^2+k_i)} \times \frac{(q)_n^2}{(q)_{n-k_1}(q)_{k_1-k_2}\cdots(q)_{k_{m-1}-k_m}(q)_{k_m}^2}$$

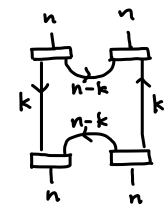


Twist formulas for (n,0) coloring of $\mathfrak{sl}_3$

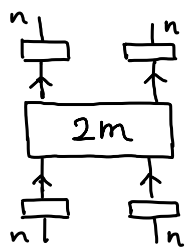
Theorem[Y. 2017] (anti-parallel case)



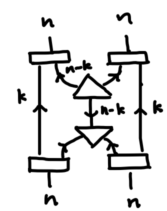
$$= q^{-\frac{2m}{3}(n^2+3n)} \sum_{n \geq k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} q^{n-k_m} q^{\sum_{i=1}^{m}(k_i^2+2k_i)} \times \frac{(q)_n^2}{(q)_{n-k_1}(q)_{k_1-k_2}\cdots(q)_{k_{m-1}-k_m}(q)_{k_m}^2}$$

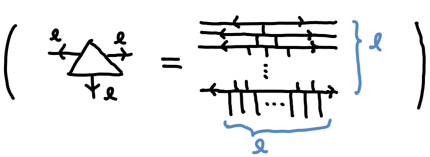


Theorem[Y. 2020]



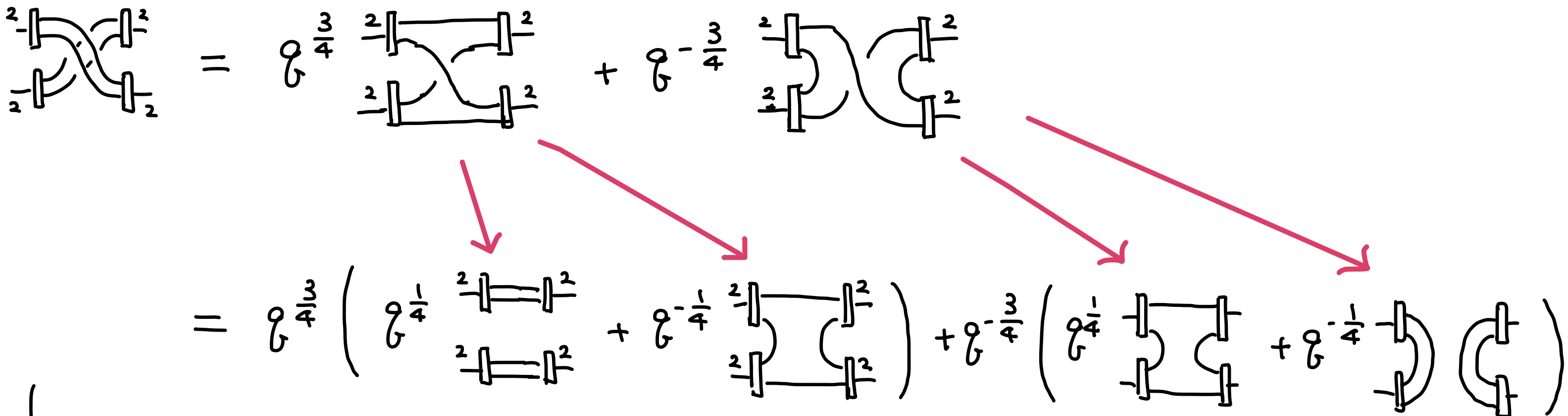
$$= q^{-\frac{m}{3}(n^2+3n)} \sum_{n \geq k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} q^{\frac{n-k_m}{2}} q^{\sum_{i=1}^{m}(k_i^2+k_i)} \times \frac{(q)_n^2}{(q)_{n-k_1}(q)_{k_1-k_2}\cdots(q)_{k_{m-1}-k_m}(q)_{k_m}^2}$$



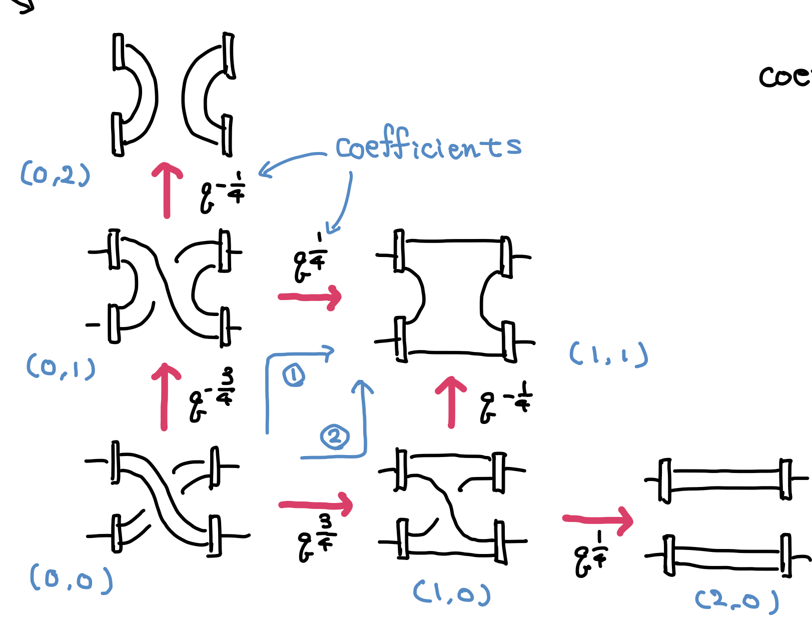


How to derive twist formulas (Y. 2017)

e.g.



by skein tree



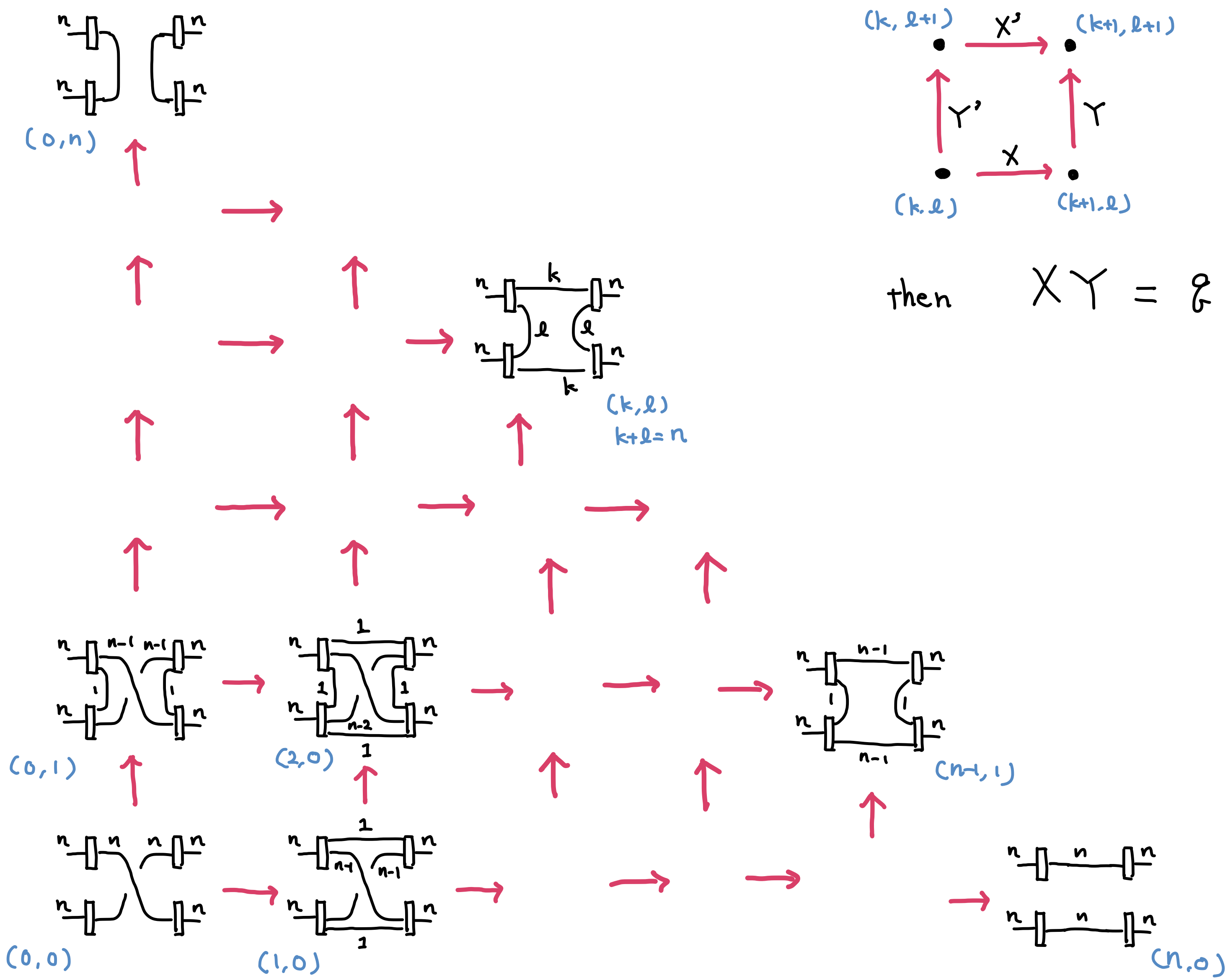
coefficient of basis web on (1,1)

$$= \sum_{\gamma:\ \text{paths from } (0,0) \text{ to } (1,1)} \ \prod_{w:\ \text{coeff. on } \gamma} w$$

$$= \underbrace{q^{-\frac{3}{4}} q^{\frac{1}{4}}}_{①} + \underbrace{q^{\frac{3}{4}} q^{-\frac{1}{4}}}_{②}$$

$\times q$

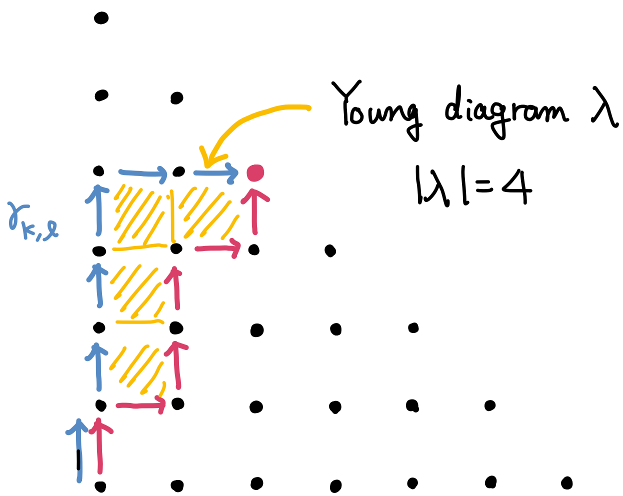
(half twist formula)

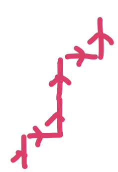
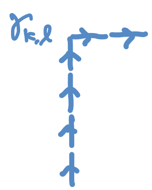


Lemma

$$\begin{tikzcd}
(k,l+1) \arrow[r, "X'"] & (k+1,l+1) \\
(k,l) \arrow[u, "Y'"] \arrow[r, "X"] & (k+1,l) \arrow[u, "Y"]
\end{tikzcd}$$

then $XY = q\, Y'X'$



coeff $\Bigg($  $\Bigg) = q^{\overset{|\lambda|}{\overset{\shortparallel}{4}}} \Bigg($  $\Bigg)$

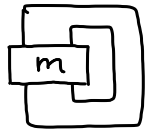
◎ coefficient of (k, l) $\quad (k + l = n)$

$$= \sum_{\substack{\gamma:\ \text{path from} \\ (0,0)\ \text{to}\ (k,l)}} \ \prod_{\substack{w:\ \text{weight} \\ \text{on}\ \gamma}} w$$

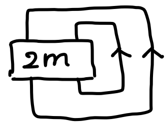
$$= \left(\prod_{\substack{w:\ \text{weight} \\ \text{on}\ \gamma_{k,l}}} w \right) \left(\sum_{\substack{\lambda:\ \text{Young diagram} \\ \#\,\text{row} \leq k \\ \#\,\text{column} \leq l}} q^{|\lambda|} \right)$$

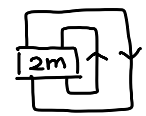
$$= \left(\prod_{\substack{w:\ \text{weight} \\ \text{on}\ \gamma_{k,l}}} w \right) \frac{(q)_n}{(q)_k (q)_{n-k}}$$

Andrews - Gordon type identities from the tail of $T(2,m)$

- $T(2,m) :=$ 

e.g. $m=2$  $m=3$ 

$T_{\overrightarrow{\rightarrow}}(2,2m) :=$ 

$T_{\overleftarrow{\rightarrow}}(2,2m) :=$ 

- $f(a,b) := \sum_{i=0}^{\infty} a^{\frac{i(i+1)}{2}} b^{\frac{i(i-1)}{2}} + \sum_{i=1}^{\infty} a^{\frac{i(i-1)}{2}} b^{\frac{i(i+1)}{2}}$: the theta series

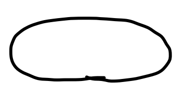
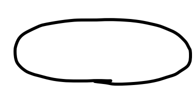
$\Psi(a,b) := \sum_{i=0}^{\infty} a^{\frac{i(i+1)}{2}} b^{\frac{i(i-1)}{2}} - \sum_{i=1}^{\infty} a^{\frac{i(i-1)}{2}} b^{\frac{i(i+1)}{2}}$: the false theta series

$\{ V_\lambda$ colored link $L \}_\lambda$

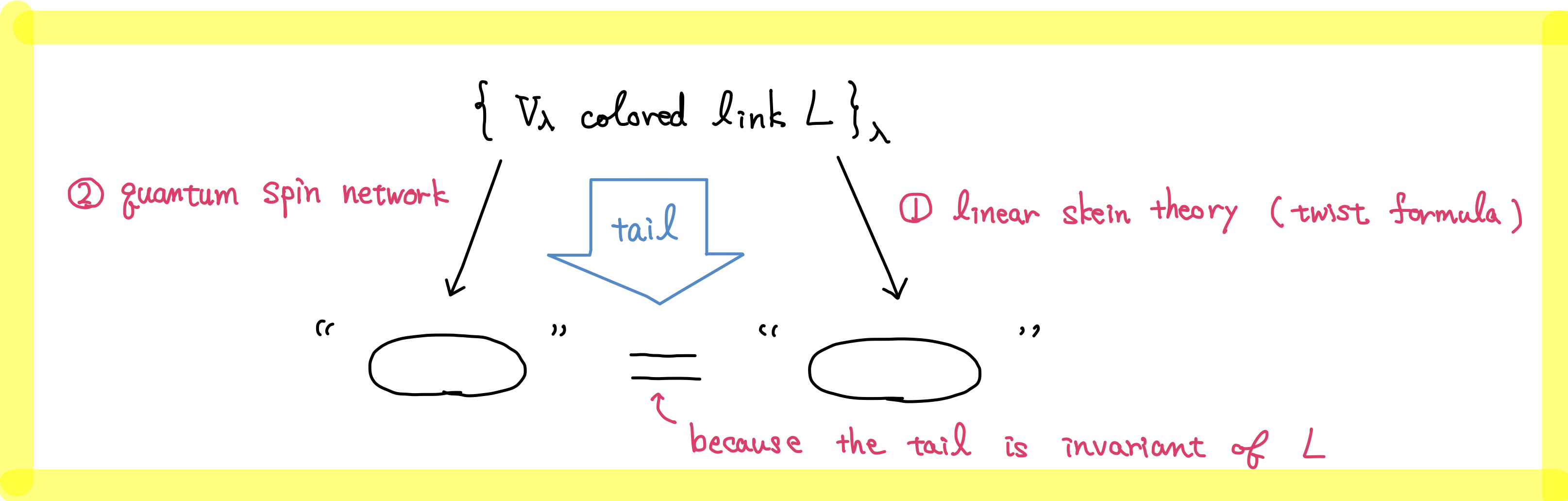
② quantum spin network

tail

① linear skein theory (twist formula)

"  " = "  "

because the tail is invariant of L



Theorem[Armond-Dasbach, 2011]

$$f(-q^{2m}, -q)/(1-q) = J^{sl_2}_{T(2,2m+1)}(q) = \frac{(q)_\infty}{1-q} \sum_{k_1 \geq k_2 \geq \cdots \geq k_{m-1} \geq 0} \frac{q^{\sum_{i=1}^{m-1} k_i^2 + k_i}}{(q)_{k_1-k_2}(q)_{k_2-k_3} \cdots (q)_{k_{m-2}-k_{m-1}}(q)_{k_{m-1}}}$$

Theorem[Hajij, 2015]

$$\Psi(q^{2m-1}, q)/(1-q) = J^{sl_2}_{T(2,2m)}(q) = \frac{(q)_\infty}{1-q} \sum_{k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} \frac{q^{-k_m}\, q^{\sum_{i=1}^{m} k_i^2 + k_i}}{(q)_{k_1-k_2}(q)_{k_2-k_3} \cdots (q)_{k_{m-1}-k_m}(q)_{k_m}^2}$$

Theorem[Y. 2018]

$$\sum_{i=0}^{\infty} q^{-2i} q^{m(i^2+2i)} \frac{(1-q^{i+1})^3(1+q^{i+1})}{(1-q)^2(1-q^2)} = J^{\mathfrak{sl}_3}_{T_{\leftrightarrows}(2,2m)}(q)$$

$$= \frac{(q)_\infty}{(1-q)(1-q^2)} \sum_{k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} \frac{q^{-2k_m} q^{\sum_{i=1}^{m} k_i^2+2k_i}}{(q)_{k_1-k_2}(q)_{k_2-k_3}\cdots(q)_{k_{m-1}-k_m}(q)^2_{k_m}}$$

a "diagonal summand" of the $\mathfrak{sl}_3$ false theta function in Bringmann - Kaszian - Milas

Theorem[Y. 2020]

$$J^{\mathfrak{sl}_3}_{T_{\rightrightarrows}(2,2m+1)}(q) = \frac{f(-q^{2m}, -q)}{(1-q)^2(1-q^2)} \qquad J^{\mathfrak{sl}_3}_{T_{\rightrightarrows}(2,2m)}(q) = \frac{\Psi(q^{2m-1}, q)}{(1-q)^2(1-q^2)}$$

$$\rightsquigarrow \quad J^{\mathfrak{sl}_3}_{T_{\rightrightarrows}(2,m)}(q) = \frac{1}{(1-q)(1-q^2)} J^{\mathfrak{sl}_2}_{T(2,m)}(q)$$

Thank you for listening