2022-07-19 南大阪 代数セミナー

Skein and cluster algebras of unpunctured surfaces for $\mathfrak{sp}_4$ (arXiv:2207.01540)

with Tsukasa Ishibashi (Tohoku Univ.)

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plan

$\Sigma = (\Sigma, \mathbb{M})$: an unpunctured marked surface

§ Main results

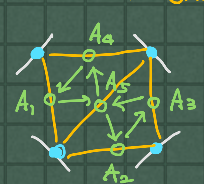
Conjecture

the "clasped" $\mathfrak{g}$-skein algebra $\quad$ the quantum cluster algebra associated with $\mathsf{S}_{\mathfrak{g}}(\mathfrak{g}, \Sigma)$

$$\mathscr{S}_{\mathfrak{g},\Sigma}[\partial^{-1}] \underset{=}{\hookrightarrow} \mathscr{A}^{q}_{\mathfrak{g},\Sigma} \underset{=}{\hookrightarrow} \mathscr{U}^{q}_{\mathfrak{g},\Sigma}$$

in Frac $\mathscr{S}_{\mathfrak{g},\Sigma}$

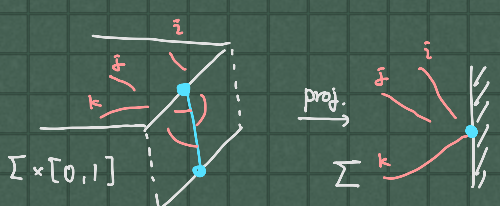
1-3-valent graph + skein relation $\quad$ cluster variables + exchange relation.



$\cup$

$\mathfrak{g}$-web

$V_{\varpi_i} \in \mathrm{FundRep}_{\mathfrak{g}}$



" The skein algebra gives a quantization of the moduli space $\mathcal{A}_{\mathfrak{g},\Sigma}$ of decorated twisted G-local systems on Σ

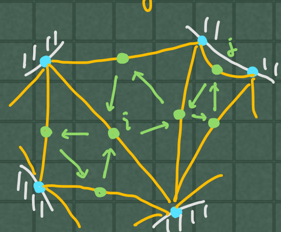
$(\mathcal{O}(\mathcal{A}^{\times}_{\mathfrak{g},\Sigma}) = \mathscr{U}_{\mathfrak{g},\Sigma})$ "

◎ Muller ('16) $\mathfrak{g} = \mathfrak{sl}_2$ $\mathfrak{sl}_2$-web = tangles on Σ
(no trivalent vertices)

$$\begin{cases} \mathcal{A}^q_{\mathfrak{sl}_2,\Sigma} \subset \mathcal{S}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] \subset \mathcal{U}_{\mathfrak{sl}_2,\Sigma} \ (\text{in } \operatorname{Frac} \mathcal{S}_{\mathfrak{sl}_2,\Sigma}) \\ \mathcal{A}^q_{\mathfrak{sl}_2,\Sigma} = \mathcal{U}^q_{\mathfrak{sl}_2,\Sigma} \end{cases}$$

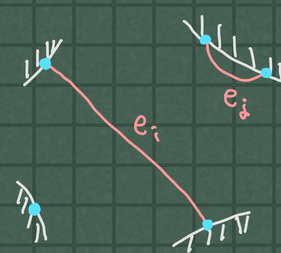
$$\rightsquigarrow \mathcal{A}^q_{\mathfrak{sl}_2,\Sigma} = \mathcal{S}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] = \mathcal{U}^q_{\mathfrak{sl}_2,\Sigma}$$

ideal triangulation Δ

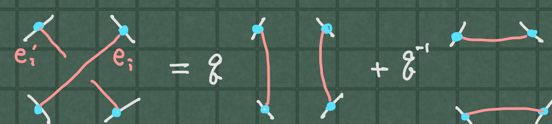


$I^\Delta_{uf} \cup I^\Delta_f$
$\ \ \rotatebox{90}{$\in$}\ i \qquad \rotatebox{90}{$\in$}\ j$

cluster variables $\{A_i \mid i \in I^\Delta\}$ $\longleftrightarrow$



q-exchange rel = skein rel



$$\begin{cases} \text{we know all cluster variables in } \mathcal{S}_{\mathfrak{sl}_2,\Sigma} \\ \text{the quantum exchange relation} \end{cases} \Rightarrow \mathcal{A}^q_{\mathfrak{sl}_2,\Sigma} \subset \mathcal{S}^q_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}]$$

= the skein relation

$\mathfrak{g}$: higher rank

Hard Realizing all cluster variables as $\mathfrak{g}$-webs in $\mathcal{S}_{\mathfrak{g},\Sigma}$

$\exists$ clusters do not come from "decorated ideal triangulations".

◎ $\mathfrak{g} = \mathfrak{sl}_3$ Ishibashi - Y. (2021)

① $\mathscr{S}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \underset{(\mathrm{I})}{\subset} \mathscr{A}^q_{\mathfrak{sl}_3,\Sigma} \subset \mathscr{U}^q_{\mathfrak{sl}_3,\Sigma} \subset \mathrm{Frac}\,\mathscr{S}_{\mathfrak{sl}_3,\Sigma}$

$\cup$ $\cup$

② "elevation-preserving" $\mathfrak{sl}_3$-webs w.r.t $\triangle$ $\overset{(\mathrm{II})}{\longmapsto}$ Laurent polynomial in $\mathcal{C}_\triangle$ with coefficients in $\mathbb{Z}_+[q^{\pm 1/2}]$

(I) the sticking trick (II) the cutting trick

◎ $\mathfrak{g} = \mathfrak{sp}_4$ Ishibashi - Y. (2022)

Theorem ① & ② in a similar way (I) (II)

difference from the $\mathfrak{sl}_3$ case :

$$\begin{cases} \mathscr{S}_{\mathfrak{sl}_3,\Sigma} : \text{a } \mathbb{Z}[q^{\pm 1/2}]\text{-algebra} \\ \mathscr{S}_{\mathfrak{sp}_4,\Sigma} : \text{a } \mathbb{Z}[q^{\pm 1/2}, 1/[2]]\text{-algebra} \end{cases}$$

- We define "the $\mathbb{Z}_q$-form $\mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}$" of $\mathscr{S}_{\mathfrak{sp}_4,\Sigma}$ and show $\mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}] \subset \mathscr{A}^q_{\mathfrak{sp}_4,\Sigma}$
- The Laurent positivity is shown in $\mathbb{Z}_+[q^{\pm 1/2}, 1/[2]]$

◎ Combine with a result in Ishibashi - Oya - Shen (2022) ($\mathscr{A} = \mathscr{U}$)

Corollary $\mathscr{S}^{\mathbb{Z}}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}] = \mathscr{A}_{\mathfrak{sp}_4,\Sigma} = \mathscr{U}_{\mathfrak{sp}_4,\Sigma} = \mathcal{O}(\mathcal{A}^\times_{\mathfrak{sp}_4})$

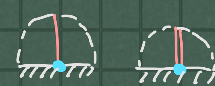
§2. the clasped sp_4-skein algebra

$R := \mathbb{Z}[v^{\pm 1/2}, 1/[2]]$ $\left([n] = \dfrac{v^n - v^{-n}}{v - v^{-1}} \right)$

$\mathcal{S}_{sp_4,\Sigma} := R\{ sp_4\text{-graphs on } \Sigma \} \big/ sp_4\text{-skein relations}$

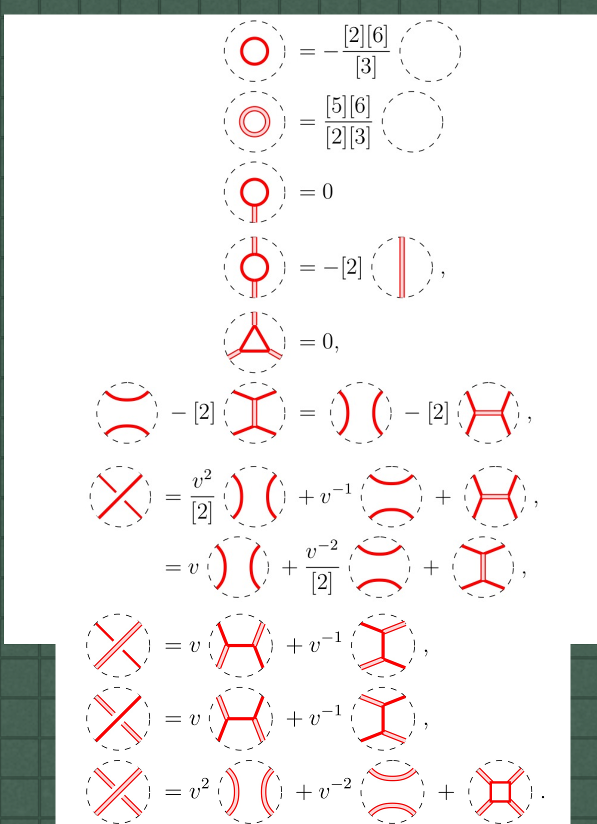
type 1 edge —— : ϖ_1
type 2 edge ═══ : ϖ_2

superposition of



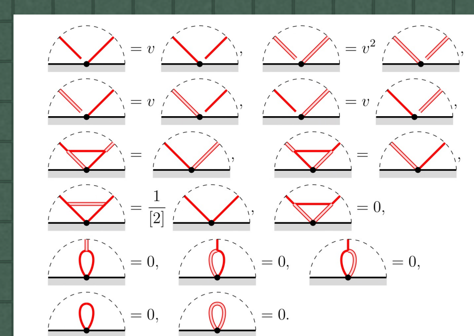
- sp_4-skein relations

- Kuperberg's "internal" skein relations

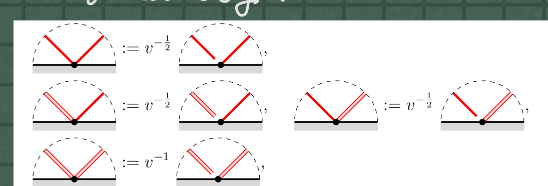


&

- "clasped" skein relation (NEW relations)



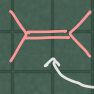
※ We use the simultaneous crossings defined by:



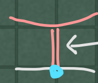
These skein relations realize the Reidemeister moves (framed ver.)

$\rightsquigarrow$  for any colors ϖ_1, ϖ_2

◎ Crossroads, rungs and legs

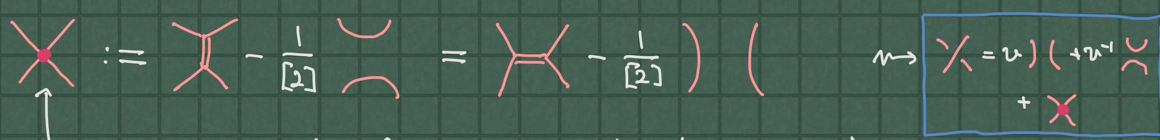


a rung (an internal type 2 edges)



a leg (a type 2 edge between a vertex & marked pt)

We define a new 4-valent vertex as



a crossroad (introduced by Kuperberg)

Definition

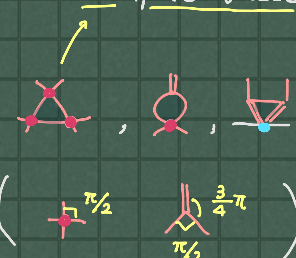
A crossroad web is an sp_4-web represented by a 1-3-4-valent graph with no rungs.

Definition

- A basis web is a flat crossroad web with no elliptic faces.

  no internal crossings
  only simultaneous crossings

  e.g.



- $\mathrm{BWeb}_{sp_4,\Sigma} := \{\text{basis webs on } \Sigma\} \subset \mathcal{S}_{sp_4,\Sigma}$

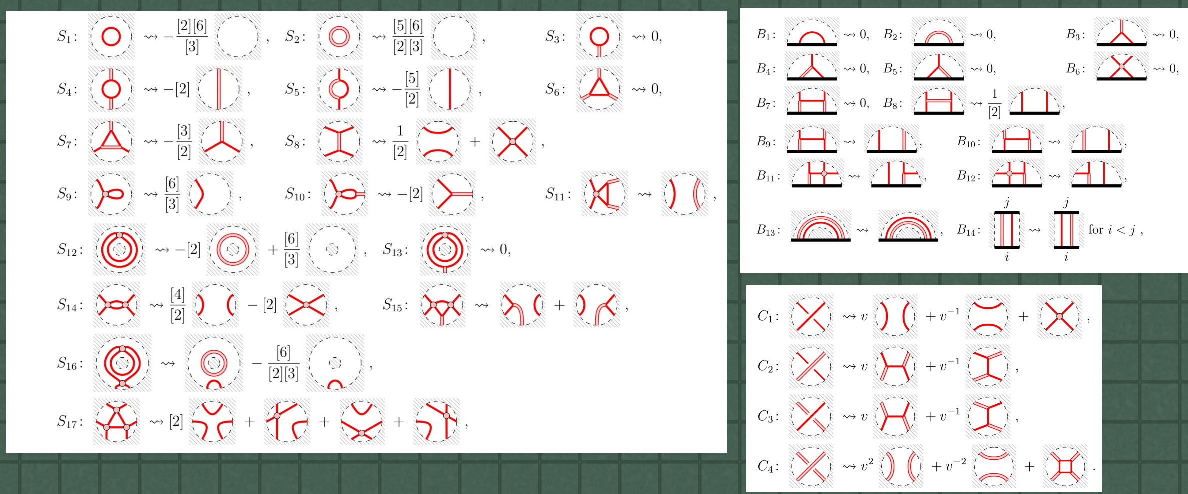
Theorem (IY)

$\mathrm{BWeb}_{sp_4,\Sigma}$ is an R-basis of $\mathcal{S}_{sp_4,\Sigma}$

proof By Sikora-Westbury's confluence theory

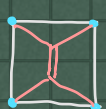
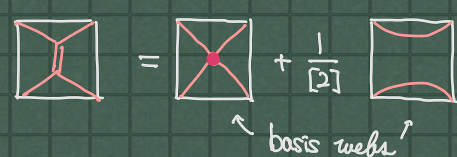
(the diamond lemma for the skein theory)

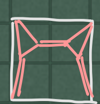
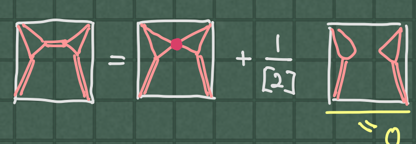
The reduction rules for sp_4-web



X : an sp_4-graph $\xrightarrow[\text{rules}]{\text{reduction}}$ $X = \sum a_i W_i$

unique   basis web

e.g.  is not a basis web. 

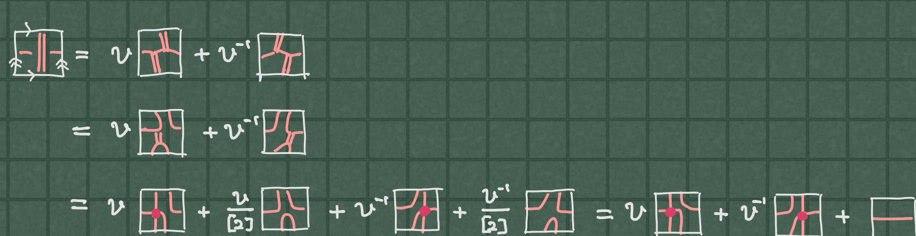
 is a basis web. 

Definition The $\mathbb{Z}_v$-form of $\mathcal{S}_{sp_4,\Sigma}$ is defined by

$$\mathcal{S}^{\mathbb{Z}_v}_{sp_4,\Sigma} := \mathbb{Z}_v\, \mathrm{BWeb}_{sp_4,\Sigma} \qquad (\mathbb{Z}_v := \mathbb{Z}[v^{\pm 1/2}])$$

Theorem (IY) $\mathcal{S}^{\mathbb{Z}_v}_{sp_4,\Sigma}$ is a $\mathbb{Z}_v$-algebra.

proof Show $^\forall G_1, {}^\forall G_2 \in \mathrm{BWeb}_{sp_4,\Sigma}$, $G_1 G_2 \in \mathbb{Z}_v \mathrm{BWeb}_{sp_4,\Sigma}$

e.g. 

§3 Construction $\mathcal{A}^{q}_{\mathrm{sp}_4,\Sigma}$ in $\mathrm{Frac}\,\mathcal{S}^{q=v}_{\mathrm{sp}_4,\Sigma}$

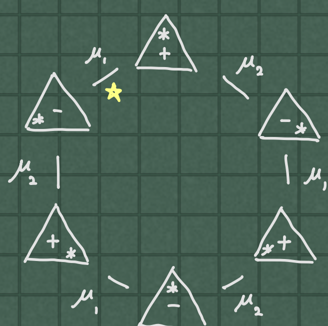
$\boxed{\text{STEP 1}}$ T = triangle 

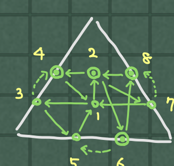
<u>Lemma</u> $\mathcal{S}^{\mathbb{Z}_q}_{\mathrm{sp}_4,T}$ is generated by

 and boundary webs  ...

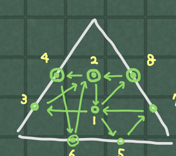
<u>Theorem</u> $\mathcal{S}^{\mathbb{Z}_q}_{\mathrm{sp}_4,T} = \mathcal{A}^{q}_{\mathrm{sp}_4,T} \leftarrow \{\text{clusters}\} = \{\text{dec. ideal triangulations}\}$

weighted quiver



 = 

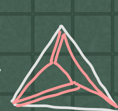
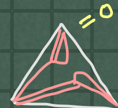
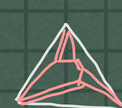
e_1 e_2

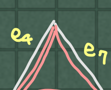
 = 

e_1 e_2

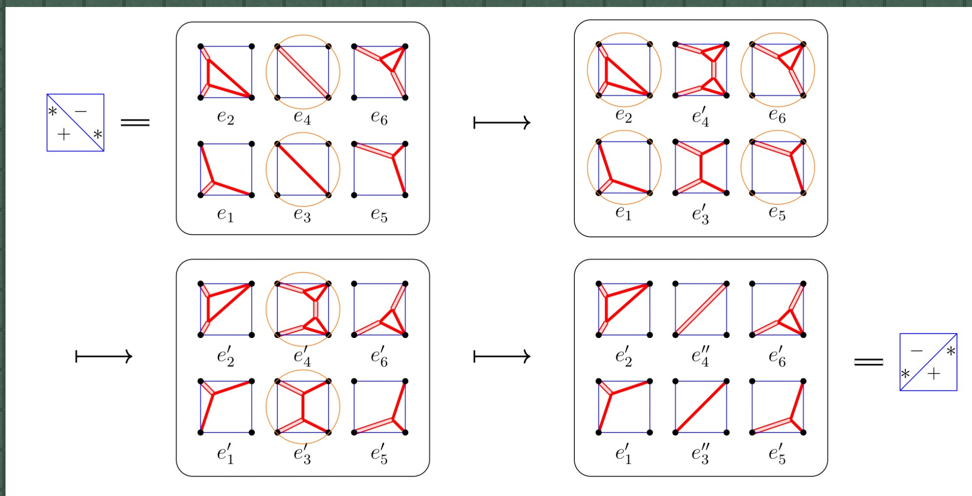
- Let us see ☆ $\mu_1 : e_1 e_1' = q^{-\frac{1}{2}}[e_2 e_3] + q^{\frac{1}{2}}[e_4 e_5 e_7]$ $e_1 =$  $e_1' =$ 

$e_1 e_1' =$  $= q^{-\frac{1}{2}}$ 

$= q^{-\frac{1}{2}} \Big(q$  $+ \dfrac{q^{-1}}{[2]}$  $+$  $\Big)$

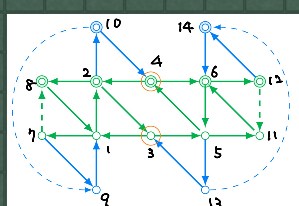
$= q^{\frac{1}{2}}$  $+ q^{-\frac{1}{2}}$ 

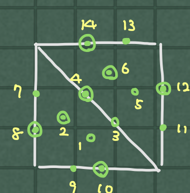
STEP 2 Check flips between decorated ideal triangulations.

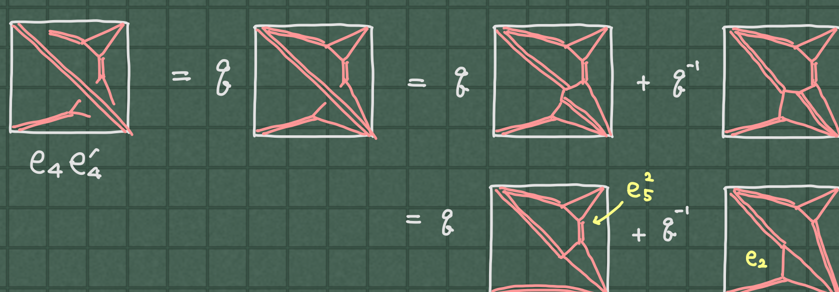


① We can confirm that the mutation sequence is realized by skein relations

e.g.





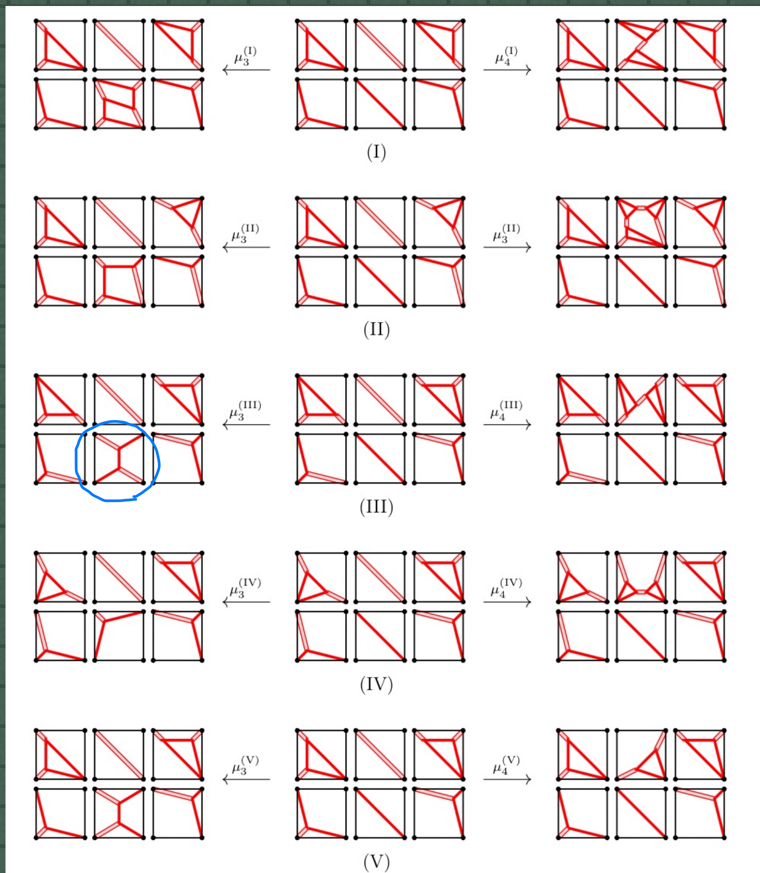


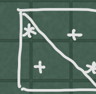
Definition

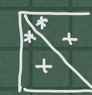
$\mathcal{A}^{q}_{\mathfrak{sp}_4,\Sigma} = \mathcal{A}_{\mathsf{S}_q(\mathfrak{sp}_4,\Sigma)}$ is the quantum cluster algebra associated with the canonical mutation class $\mathsf{S}_q(\mathfrak{sp}_4,\Sigma)$ containing $\{Q^{\Delta}\}$ all decorated ideal triangulation Δ.

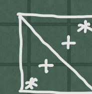
Remark $\forall \Delta, \Delta'$: decorated ideal triangulation $\Delta \leftrightarrow \cdots \leftrightarrow \Delta'$ $\exists$ mutation sequence

Other $\mathfrak{sp}_4$-webs in $\mathcal{A}_{\mathfrak{sp}_4,\Sigma}$



(I) = 

(II) = 

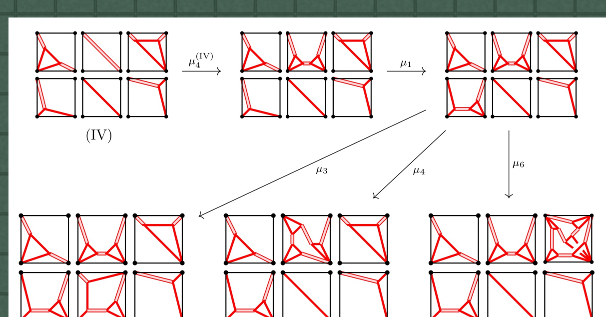
(III) = 

(IV) = 

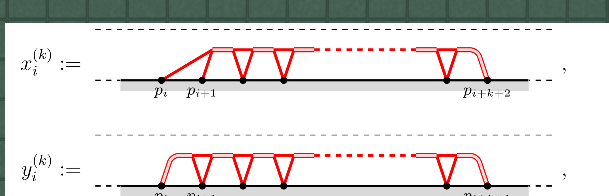
(V) = 

All matrix elements of a simple Wilson line  appear in these sequence. [c.f. Ishibashi - Oya - Shen '22]

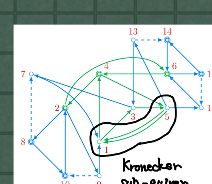
More examples



An infinite sequence



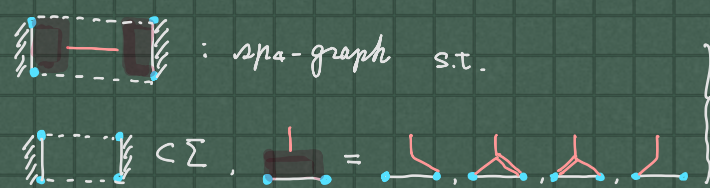
$$x_0^{(k+4)} x_0^{(k)} = v^{\bullet}\, y_{k+2}^{(3)} + v^{\blacktriangle} (x_2^{(k+2)})^2$$



§4. $\mathscr{S}_{sp_4,\Sigma}[\partial^{-1}] \subset \mathscr{A}^{q}_{sp_4,\Sigma}$

- Consequence of "§3. $\mathscr{A}^{q}_{sp_4,\Sigma} \subset \operatorname{Frac} \mathscr{S}_{sp_4,\Sigma}$"

(1) $\mathscr{A}^{q}_{sp_4,\Sigma} \subset \operatorname{Frac} \mathscr{S}_{sp_4,\Sigma}$

(2) $\mathrm{SimpWil}^{\varpi_i}_{sp_4,\Sigma} := \{$  : sp_4-graph s.t. $\subset \Sigma$ $\}$

"simple Wilson lines" colored by ϖ_i

$\boxed{\text{Show } \mathscr{S}^{\mathbb{Z}_q}_{sp_4,\Sigma}[\partial^{-1}] \text{ is generated by } \mathrm{SimpWil}^{\varpi_i}_{sp_4,\Sigma} \text{ (and } \mathscr{S}_{sp_4,\Sigma} \text{ is an Ore domain)}}$ $\rightsquigarrow$ $\mathscr{S}^{\mathbb{Z}_q}_{sp_4,\Sigma}[\partial^{-1}] \subset \mathscr{A}^{q}_{sp_4,\Sigma}$

<u>Theorem</u> (IY.) $\mathscr{S}_{sp_4,\Sigma}[\partial^{-1}]$ is an Ore domain.

<u>Sketch of proof</u>

$$\begin{array}{ccc}
\mathscr{S}_{sp_4,\Sigma}[\partial^{-1}] & & \\
\downarrow \cong & & \\
\mathscr{S}^{\text{stated, rd}}_{sp_4,\Sigma} & \hookrightarrow & \bigotimes_{T\in\Delta} \mathscr{S}^{\text{stated, red}}_{sp_4,T} \\
& & \downarrow \cong \\
& & \bigotimes_{T\in\Delta} \mathscr{S}_{sp_4,T}[\partial^{-1}] = \bigotimes_{T\in\Delta} \mathscr{A}^{q}_{sp_4,T}(\mathcal{R})
\end{array}$$

[IY, in preperation]

$\downarrow\cong$: the state-clasp correspondence (c.f. sl_2: Lê-Yu)

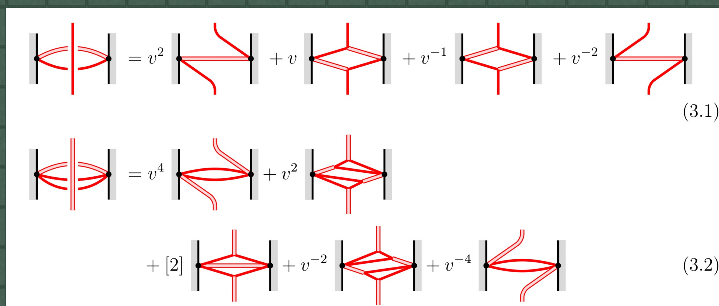
$\hookrightarrow$: the splitting homomorphism

(c.f. sl_2 : T.T.Q. Lê, sl_3 : Higgins, sl_n : Lê-Sikora)

<u>Corollary</u>. $\mathscr{S}^{\mathbb{Z}_q}_{sp_4,\Sigma}[\partial^{-1}] \subset \operatorname{Frac} \mathscr{S}_{sp_4,\Sigma}$

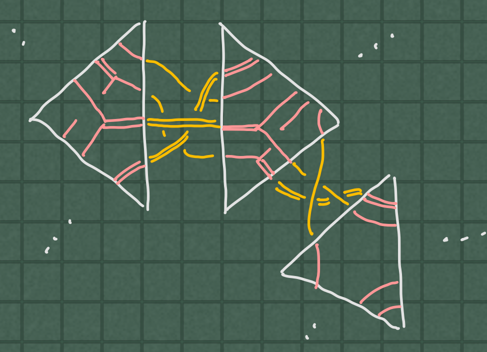
◎ The Cutting trick & the sticking trick.

Lemma (the cutting trick)



Remark
- The coefficents are positive
- $\mathcal{S}^{\mathbb{Z}_q}_{sp_4,\Sigma} \subset \mathcal{U}_{sp_4,\Sigma}$

$\rightsquigarrow$ Laurent positivity for "elevation preserving webs".

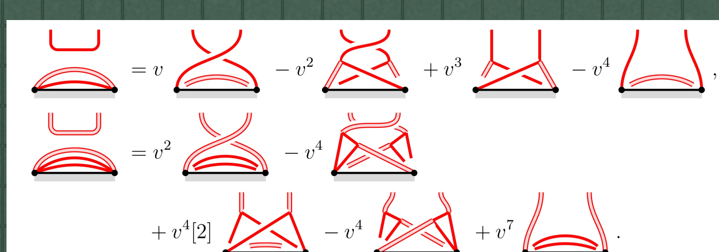


$\in \mathcal{A}^q_{sp_4,\Sigma} \otimes R$

$\uparrow$

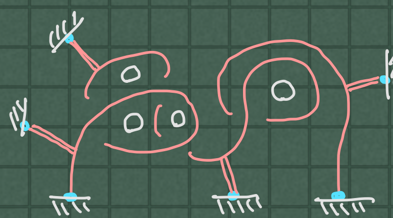
$\mathbb{Z}_q[1/[2]]$

Lemma (the sticking trick)



Proposition $\mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}$ is generated by

$\begin{cases} \bullet \text{ "descending loops \& arcs with/without legs" of type 1} \\ \bullet \text{ simple loops/arcs of type 2} \end{cases}$



proof. Use a filtration by the "number" of crossings and crossroads

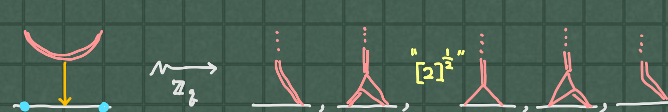
localization by boundary webs

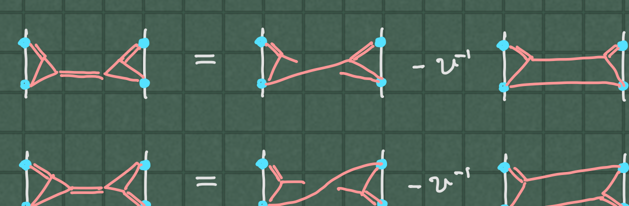
Theorem $\mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}]$ is generated by $\mathrm{SimpWil}^{\mathfrak{S}}_{\mathfrak{sp}_4,\Sigma}$

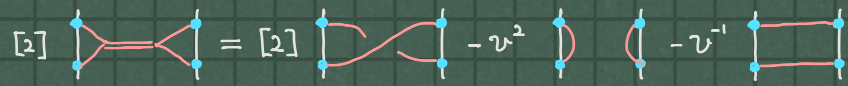
for Σ with $\#\{\text{marked points}\} \geq 2$.

proof ⓪ arcs of type 2

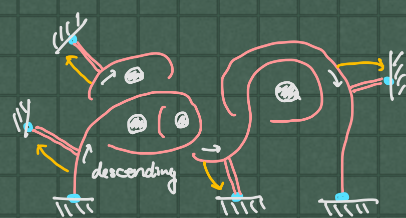
① loops of type 2 ① the sticking trick for type 2

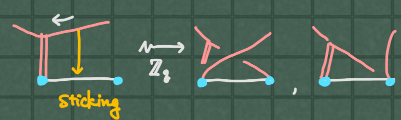




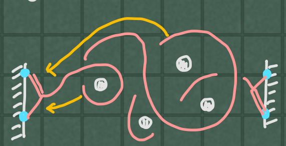


11 descending curves with / without legs

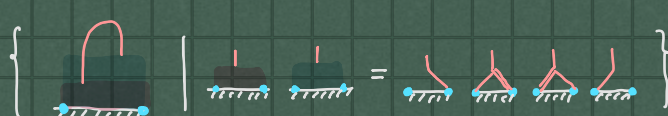


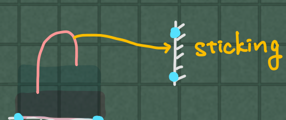


a $\mathbb{Z}_q$-polynomial in

$\{$  descending arc of with $n \leq 2$ legs $\}$

a $\mathbb{Z}_q$-polynomial in





use $\#\{\text{marked points}\} \geq 2$

a $\mathbb{Z}_q$-polynomial in

$$\mathrm{SimpWil}^{\infty,}_{sp_4,\Sigma}$$

∎

<u>Corollary</u> $\mathcal{S} = \mathcal{A} = \mathcal{U} = \mathcal{O}$ at $q = 1$

§5 Characterization of cluster variables

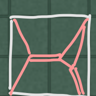
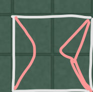
Conjecture $\quad \mathrm{EWeb}_{\mathfrak{sp}_4,\Sigma} \setminus (\mathrm{EWeb}_{\mathfrak{sp}_4,\Sigma})^{DT} = \mathrm{Tree}_{\mathfrak{sp}_4,\Sigma} = \mathrm{CV}_{\mathfrak{sp}_4,\Sigma}$

invariant under Donaldson-Thomas transformation

Definition $\quad G \in \mathrm{BWeb}_{\mathfrak{sp}_4,\Sigma}$ is an elementary web if

- G is indecomposable and
- $\forall S \subset \{\text{rungs of } G\}$, $G \setminus S = 0$ in $\mathcal{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}$

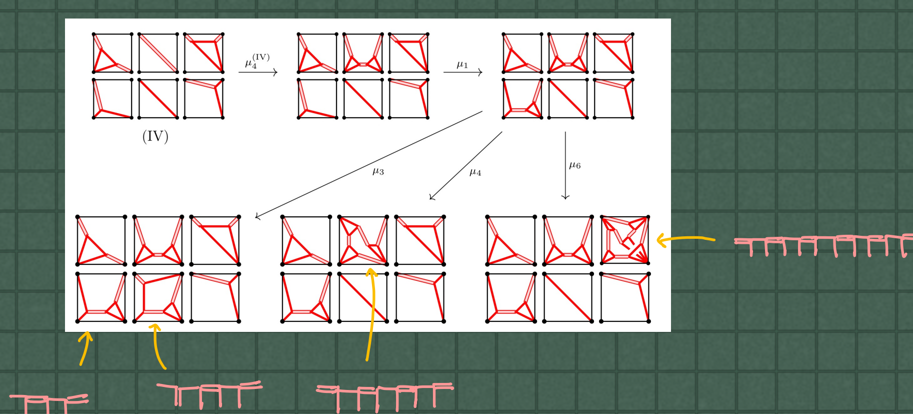
$\mathrm{EWeb}_{\mathfrak{sp}_4,\Sigma} := \{\text{elementary webs on } \Sigma\}$

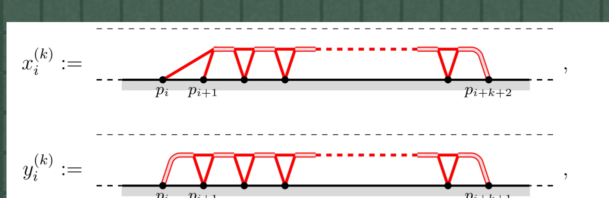
e.g.  $\rightsquigarrow$  $= 0$

Definition

$G \in \mathrm{EWeb}_{\mathfrak{sp}_4,\Sigma}$ is tree-type $:\Leftrightarrow$ $\exists\, \tilde{G}$: tree s.t. $G = q^{\circledast}\tilde{G}$

e.g.





Thank you