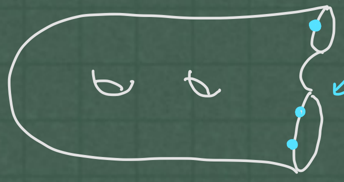
20220518
大阪大学トポロジー
セミナー

Skein and cluster algebras of unpunctured marked surfaces for $\mathfrak{sp}_4$.

joint work with Tsukasa Ishibash (Tohoku Univ.)

Wataru Yuasa (OCAMI)

$\Sigma =$ 

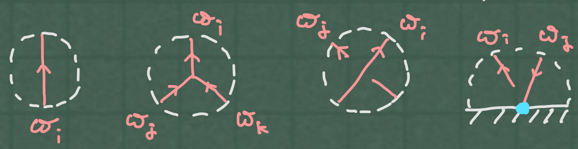
← special points $\mathbb{M}$

: (unpunctured) marked surface

☆ $\exists$ an ideal triangulation Δ

"$\mathfrak{g}$-web": (ori.) tangled trivalent graphs on Σ

colored by $V_{\varpi_i} \in \mathrm{Fund\,Rep}^{\mathrm{fd}}(\mathfrak{g})$



etc.

① Compare two non-commutative algebras

q-Laurent phenomenon ↓ upper cluster ↓

$$\mathscr{S}_{\mathfrak{g},\Sigma}[\partial^{-1}] \longleftrightarrow \mathscr{A}^{q}_{\mathfrak{g},\Sigma} \subset \mathscr{U}^{q}_{\mathfrak{g},\Sigma}$$

(boundary-localized) clasped $\mathfrak{g}$-skein algebra of Σ

quantized ring of the moduli space of decorated twisted G-local systems on Σ

quantum cluster algebra assciated with mutation class of q-seeds $s_q(\mathfrak{g},\Sigma)$

$\mathfrak{g}$-webs / skein rel.

cluster variables / exchange rel.

Conjecture

$$\mathscr{S}_{\mathfrak{g},\Sigma}[\partial^{-1}] \underset{\text{sticking trick}}{\overset{\supseteq}{=}} \mathscr{A}^{q}_{\mathfrak{g},\Sigma} = \mathscr{U}^{q}_{\mathfrak{g},\Sigma} \subset \mathrm{Frac}\,\mathscr{S}_{\mathfrak{g},\Sigma}$$

skew field of fractions

② Describe structures of $\mathcal{A}^{q}_{\mathfrak{g},\Sigma}$ by $\mathfrak{g}$-webs in $\mathcal{S}_{\mathfrak{g},\Sigma}[\partial^{-1}]$

Conjecture

$$\underbrace{\mathrm{EWeb}_{\mathfrak{g},\Sigma}}_{\text{elementary webs}} \setminus (\mathrm{EWeb}_{\mathfrak{g},\Sigma})^{DT} = \underbrace{\mathrm{Tree}_{\mathfrak{g},\Sigma}}_{\text{tree-type elementary webs}} = \underbrace{\mathrm{CV}_{\mathfrak{g},\Sigma}}_{\text{cluster variables}}$$

③ Laurent positivity for "elevation-preserving" web

Conjecture $x \in \mathcal{S}_{\mathfrak{g},\Sigma}$: elevation-preserving $\mathfrak{g}$-web

$$\begin{tikzcd} \mathcal{S}_{\mathfrak{g},\Sigma} \arrow[r, "\text{cutting trick}"] & \mathcal{U}^{q}_{\mathfrak{g},\Sigma} \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ x \arrow[r, mapsto] & x = \sum_{i\in I} a_i f_i \end{tikzcd}$$

: Laurent expansion in a cluster

then $a_i \in \mathbb{Z}_+[q^{\pm 1/2}]$

Related works

- $\mathfrak{sl}_2$: Muller (2016, ①)
  $\mathfrak{sl}_3$: Frohman-Sikora (2021, $\mathcal{S}_{\mathfrak{sl}_3,\Sigma}$)
  Fomin-Pylyavskyy (2016, ③ at $q=1$)

Our related works

rank 2
- $\mathfrak{sl}_3$: Ishibashi - Y. (2021)
- $\mathfrak{sp}_4$: Ishibashi - Y. (in preperation) : today's talk
- $\mathfrak{g}_2$: Ishibashi - Y. (in progress)

* $\mathfrak{sl}_2$ with coefficients : [Ishibashi - Kano - Y.] (in preperation)
$\rightsquigarrow$ cluster algebras with coefficients

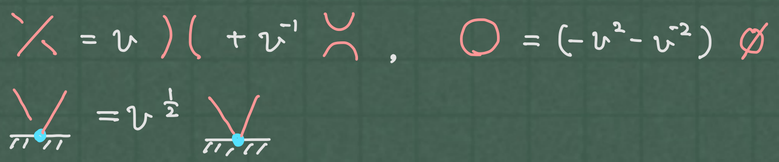
* State - clasp correspondence for $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3, \mathfrak{sp}_4$
$\rightsquigarrow$ stated skein algebras
[Ishibashi -Y.] (in preperatim)

◎ Overview of Muller's work ($\mathfrak{g} = \mathfrak{sl}_2$)

$\mathcal{S}_{\mathfrak{sl}_2,\Sigma}$: the Kauffman bracket skein algebra of $(\Sigma, \mathbb{M})$

- $\mathfrak{sl}_2$-web : tangled arcs on Σ. color : 2-dim. irrep. V_2

◎ skein rel :



$\mathcal{F}$

$\cup$

$\mathcal{A}^{q}_{\mathfrak{sl}_2,\Sigma}$: the quantum cluster algebra obtained from $s_q(\mathfrak{sl}_2, \Sigma)$

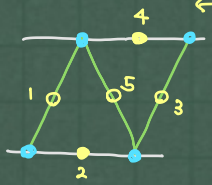
◎ clusters $\overset{1:1}{\longleftrightarrow}$ ideal triangulations of Σ

$\downarrow$

q-commuting cluster variables $\{A_i\}_i$

$\wr$

q-torus



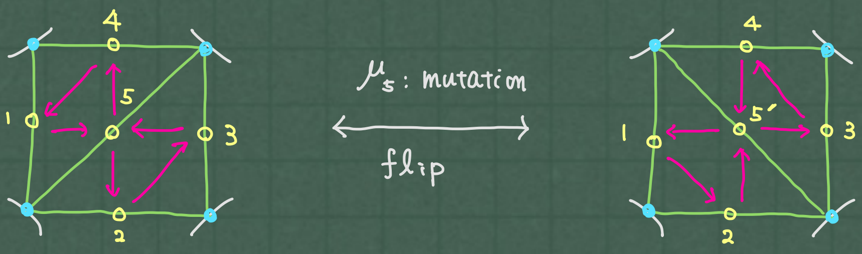
← frozen variable : invertible

← unfrozen variable

$A_i A_j = q^{\bullet} A_j A_i$

◎ quantum exchange relations

$$\underline{A_5 A_5' = q A_1 A_3 + q^{-1} A_2 A_4}$$



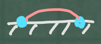
Theorem (Muller 2016) For $\Sigma = (\Sigma, \mathbb{M})$ with $\# \mathbb{M} \geq 2$

- $\mathscr{S}_{\mathfrak{sl}_2,\Sigma}$ is a Ore domain $\rightsquigarrow$ $\mathscr{S}_{\mathfrak{sl}_2,\Sigma} \hookrightarrow \mathrm{Frac}\, \mathscr{S}_{\mathfrak{sl}_2,\Sigma}$
- $\mathscr{A}^q_{\mathfrak{sl}_2,\Sigma} \subset \mathscr{S}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] \subset \mathscr{U}^q_{\mathfrak{sl}_2,\Sigma}$ in $\mathrm{Frac}\, \mathscr{S}_{\mathfrak{sl}_2,\Sigma}$.
- $\mathscr{A}^q_{\mathfrak{sl}_2,\Sigma} = \mathscr{U}^q_{\mathfrak{sl}_2,\Sigma}$

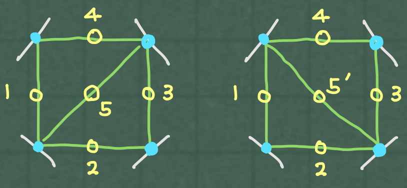
$$\rightsquigarrow \mathscr{A}^q_{\mathfrak{sl}_2,\Sigma} = \mathscr{S}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] = \mathscr{U}^q_{\mathfrak{sl}_2,\Sigma}$$

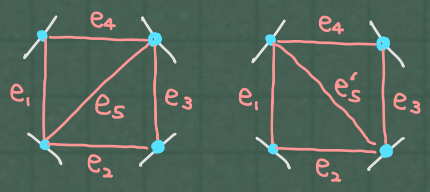
proof

(0) Ore domain : detailed argument about basis of $\mathscr{S}_{\mathfrak{sl}_2,\Sigma}$

localization at boundary webs 

(1)
$$\begin{tikzcd} \mathscr{A}_{\mathfrak{sl}_2,\Sigma} \arrow[r, hook] & \mathscr{S}_{\mathfrak{sl}_2,\Sigma}[\underset{\sim}{\partial^{-1}}] \subset \mathrm{Frac}(\mathscr{S}_{\mathfrak{sl}_2,\Sigma}) \\ A_i \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & e_i \arrow[u, phantom, "\cup"] \end{tikzcd}$$

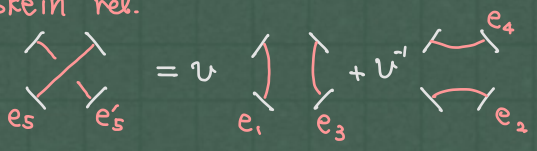




q-exchange rel.

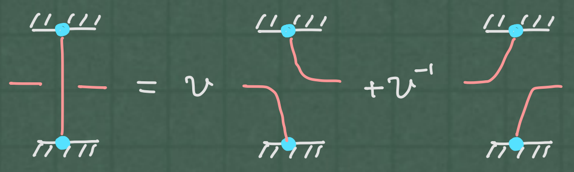
$A_5 A_5' = q A_1 A_3 + q^{-1} A_2 A_4$ $\longleftrightarrow$ compatible

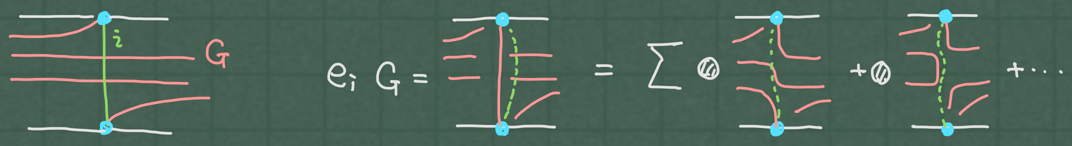
skein rel.



(2) $\mathscr{S}_{\mathfrak{sl}_2,\Sigma} \xhookrightarrow[\text{cutting trick}]{} \mathscr{U}^q_{\mathfrak{sl}_2,\Sigma}$: Laurent expansion of $\mathfrak{sl}_2$-webs

Lem (the cutting trick for $\mathfrak{sl}_2$)





reduction of intersection points

$$\xrightarrow[\text{on ideal triangulation}]{\text{cut along}} \Big(\prod_{i\in\Delta} e_i^{n_i}\Big) G = \text{polynomial of } \underline{\text{webs in triangles}}$$

e_i's

G = a Laurent polynomial in a cluster

! (3) $\mathcal{A}^q_{\mathfrak{sl}_2,\Sigma} = \mathcal{U}^q_{\mathfrak{sl}_2,\Sigma}$ from a criterion of $\mathcal{A} = \mathcal{U}$ (the Banff algorithm)

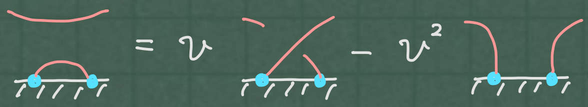
theory of cluster algebras

$$\rightsquigarrow \mathcal{A}^q_{\mathfrak{sl}_2,\Sigma} = \mathcal{S}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] = \mathcal{U}^q_{\mathfrak{sl}_2,\Sigma}$$

⊘ Another proof : $\mathcal{A}^q_{\mathfrak{sl}_2,\Sigma} = \mathcal{S}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}]$

(4) Construct an inclusion $\mathcal{S}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] \hookrightarrow \mathcal{A}^q_{\mathfrak{sl}_2,\Sigma}$

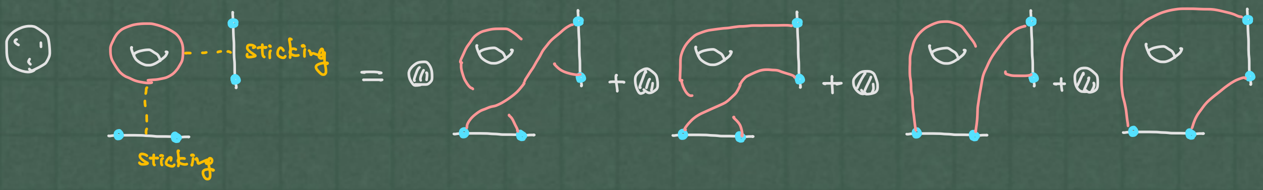
<u>Lem</u> (the sticking trick for $\mathfrak{sl}_2$)



<u>Prop</u> $\underline{\mathcal{S}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}]}$ is generated by <u>simple arcs</u> ← cluster variables in $\mathcal{A}^q_{\mathfrak{sl}_2,\Sigma}$

$\mathbb{Z}_q$-basis = {multi-curves}

∋ simple loop



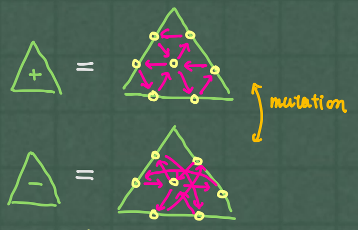
◎ Can we generalize (0) ~ (3) to rank 2 cases ?

(0) basis of $\mathscr{S}_{\mathfrak{g},\Sigma}$ is beyond control of intersection number.

$\underset{\text{replace}}{\rightsquigarrow}$ state-clasp correspondence [Ishibasi-Y. in prep.]

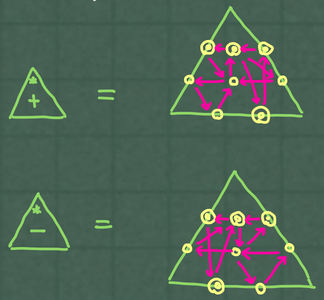
(1) ideal triangulations are "decorated"

$\mathfrak{sl}_3$:



2 clusters

$\mathfrak{sp}_4$:



(3 rotations) × (signs) = 6 clusters

∃ clusters associated with decorated ideal triangulation

e.g. $\mathfrak{sl}_3$:

: realized by 4 mutations

$\underline{\underline{(2)}}$ the cutting trick works $\rightsquigarrow$ $\mathfrak{g}$-webs in a triangle ?

(3) $\mathscr{A}_{\mathfrak{g},\Sigma}^{q=1} = \mathscr{U}_{\mathfrak{g},\Sigma}^{q=1}$ [Ishibashi - Oya - Shen 2022]

quantum case : unknown

$\underline{\underline{(4)}}$ the sticking trick works $\rightsquigarrow$ generators of $\mathscr{S}_{\mathfrak{g},\Sigma}[\partial^{-1}]$?

◎ Strategy

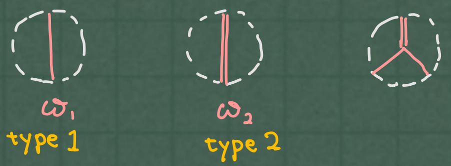
(0) & (4) $\rightsquigarrow$ $\mathscr{S}_{\mathfrak{g},\Sigma}[\partial^{-1}] \subset \mathscr{A}^{q}_{\mathfrak{g},\Sigma}$

(2) $\rightsquigarrow$ Laurent expression of a $\mathfrak{g}$-web $\rightsquigarrow$ $\mathscr{S}_{\mathfrak{g},\Sigma}[\partial^{-1}] \subset \mathscr{U}^{q}_{\mathfrak{g},\Sigma}$

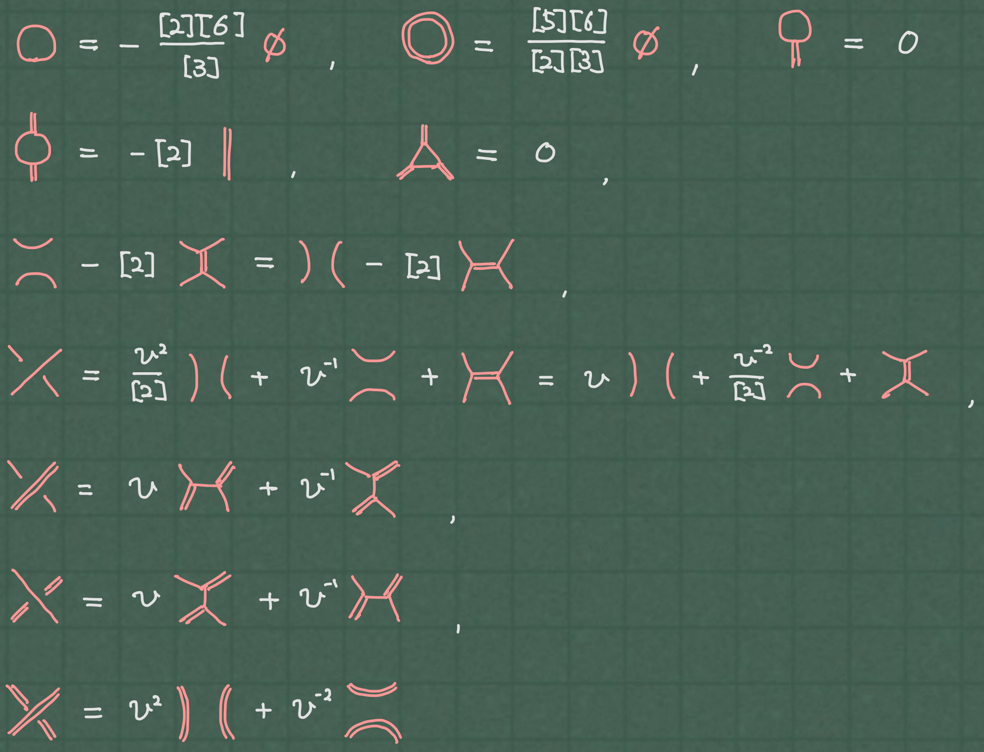
\+ elevation-preserving condition $\rightsquigarrow$ positive coefficients

§ the clasped sp_4-skein algebra

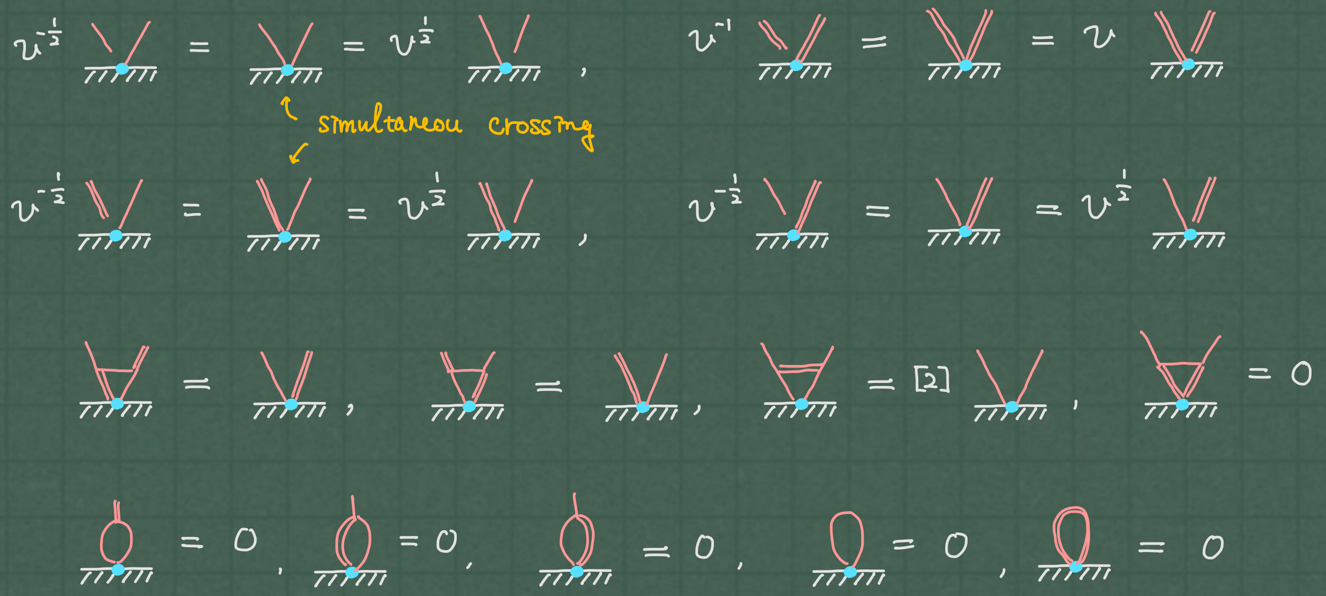
◎ sp_4-web :



◎ interior skein rel. [Kuperberg 1996]



◎ clasped skein relations [Ishibashi - Y.]



Def (the clasped $\mathfrak{sp}_4$-skein algebra)

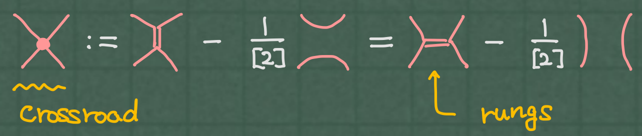
$$\mathcal{S}_{\mathfrak{sp}_4,\Sigma} := R\{\mathfrak{sp}_4\text{-webs on } \Sigma\} \Big/ \begin{array}{l}\text{interior}\\ \text{\& clasped skein rel's.}\end{array}$$

where $R = \mathbb{Z}[v^{\pm 1/2}, \frac{1}{[2]}]$

- R-basis

Def (crossroads, rungs)

- A crossroad is a 4-valent vertex defined by



- A rung is an internal edge of type 2.

(flat) ladder web (flat) crossroad web

no internal crossings
only have simultaneous crossings

Def A basis web is a flat crossroad webs with no "elliptic faces".

etc.

Thm BWeb_Σ is an R-basis of $\mathcal{S}_{\mathfrak{sp}_4,\Sigma}$

where $\mathrm{BWeb}_\Sigma := \{\text{basis webs on } \Sigma\}$

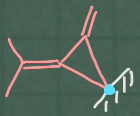
⑪ elementary webs

Def G : a flat ladder web

- an elementary web G satisfies

  $\left\{\begin{array}{l} \bullet\ {}^{\forall}S \subset \{\text{rungs of } G\},\ G_S = 0 \text{ in } \mathscr{S}_\Sigma \\ \quad \text{where } G_S \text{ is obtained by removing } S \text{ from } G. \\ \bullet\ G \text{ is indecomposable to } \prod_i G_i \quad (G_i \in \mathrm{BWeb}) \\ \bullet\ \text{the associated crossroad web of } G \text{ is in } \mathrm{BWeb} \end{array}\right.$

- an elementary web G is tree-type

  $:\Leftrightarrow\ G = v \circ {}^{\exists}G'$ s.t. $\left\{\begin{array}{l} \text{underlying graph of } G' \text{ is tree} \\ {}^{\forall}\text{rung of } G' \text{ is} \end{array}\right.$ 

Conj. $\mathrm{EWeb}_\Sigma \setminus (\mathrm{EWeb}_\Sigma)^{DT} = \mathrm{Tree}_\Sigma \ (= CV_\Sigma)$

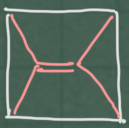
↙ invariant under Donaldson-Thomas transformations

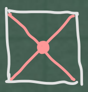
$\mathrm{EWeb}_\Sigma := \{\text{elementary webs on } \Sigma\}$

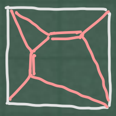
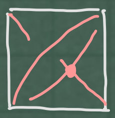
$\cup$

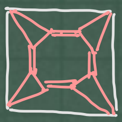
$\mathrm{Tree}_\Sigma := \{\text{tree-type elementary webs on } \Sigma\}$

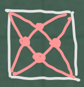
e.g.

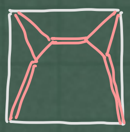
 $\notin$ EWeb

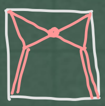
↳  $\in$ BWeb

 =  $\notin$ EWeb

 $\in (\mathrm{EWeb})^{DT}$

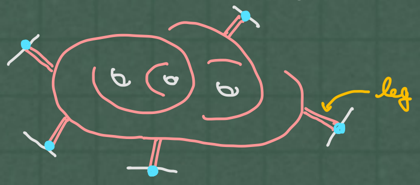
↳  $\in$ BWeb

 $\in$ Tree

↳  $\in$ BWeb

◎ generating set of $\mathcal{S}_{sp_4,\Sigma}$

$\mathrm{Desc}_\Sigma := \{$ descending knots / arcs with/without legs $\}$

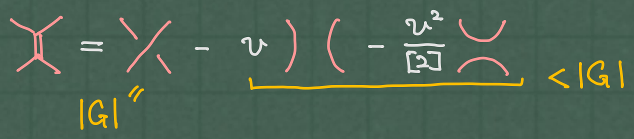


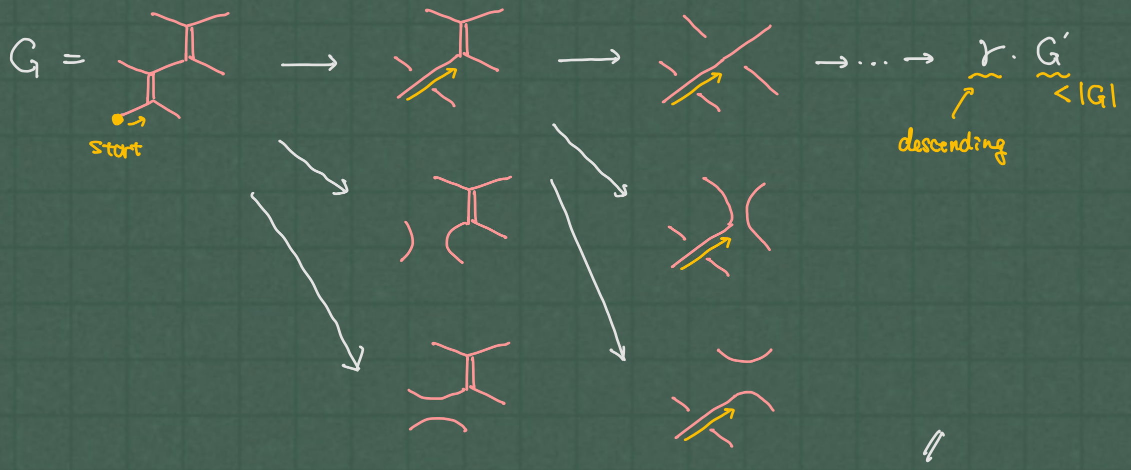
Prop An R-algebra $\mathcal{S}_{sp_4,\Sigma}$ is generated by Desc_Σ

proof By induction on the complexity $|G|$ of an sp_4-web G.

$|G| \; {=} \;$ # rungs + # internal crossings

Use :

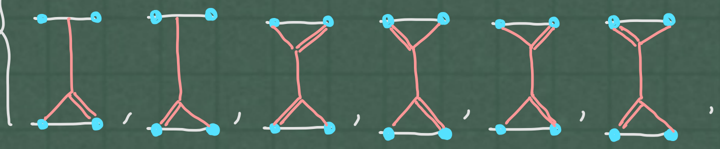




Theorem For any $\Sigma = (\Sigma, \mathbb{M})$ with $\#\mathbb{M} \geq 2$,

an R-algebra $\mathcal{S}_{sp_4,\Sigma}[\partial^{-1}]$ is generated by $\mathrm{SimpWil}_\Sigma$.

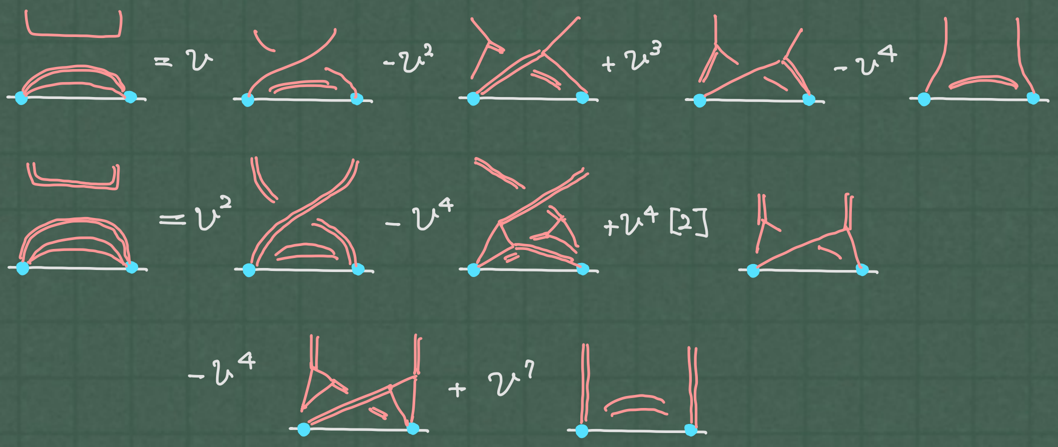
$\mathrm{SimpWil}_\Sigma := \{$

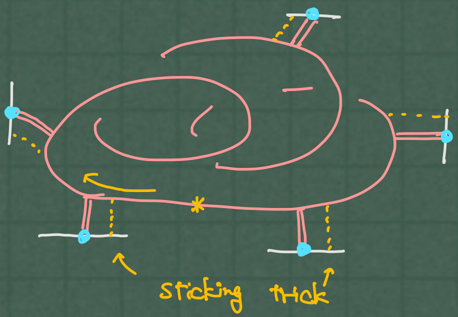


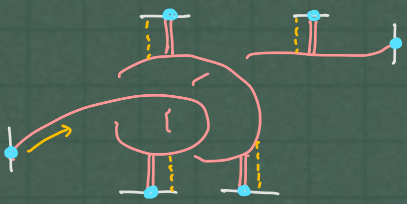
simple arcs of type 1 $\}$

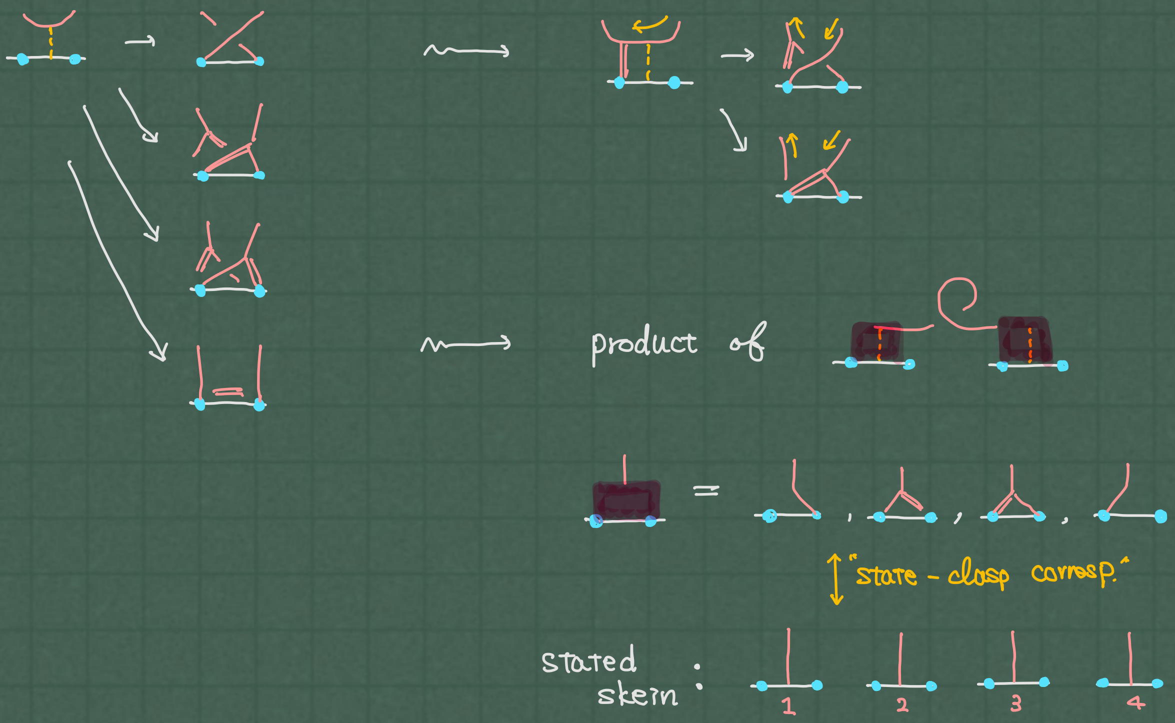
proof Apply the sticking trick to Desc_Σ.

Lem. (the sticking trick for $\mathfrak{sp}_4$)







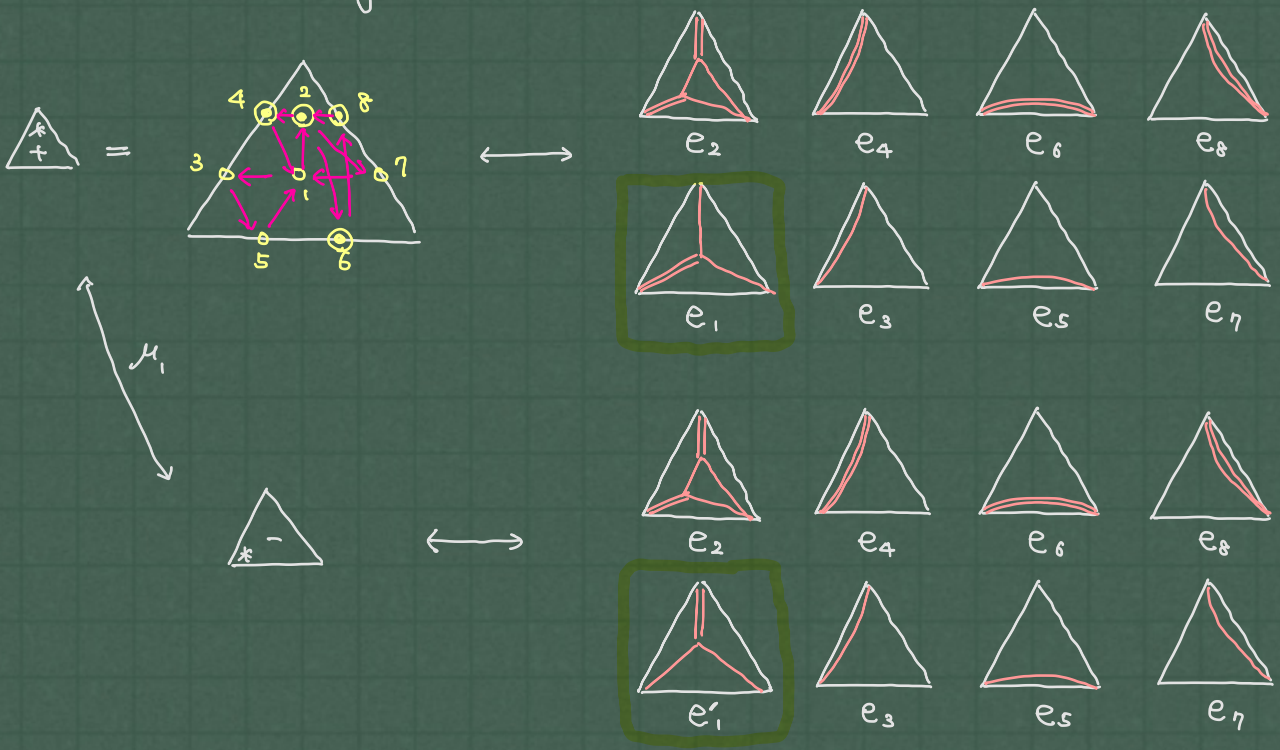


▨ If $\mathrm{SimpWil}_\Sigma$ is the set of cluster variables + skein / exchange rel. are compatible.

$\Rightarrow \mathcal{S}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}] \subset \mathcal{A}^q_{\mathfrak{sp}_4,\Sigma}$

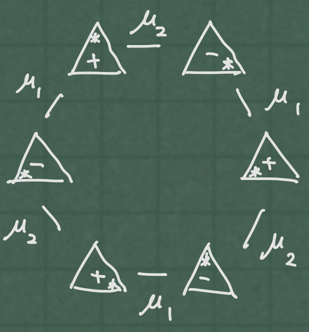
§ Correspondence between $\mathscr{S}_{\mathfrak{sp}_4,\Sigma}$ and $\mathscr{A}_{\mathfrak{sp}_4,\Sigma}$

◎ $\Sigma = T$: triangle



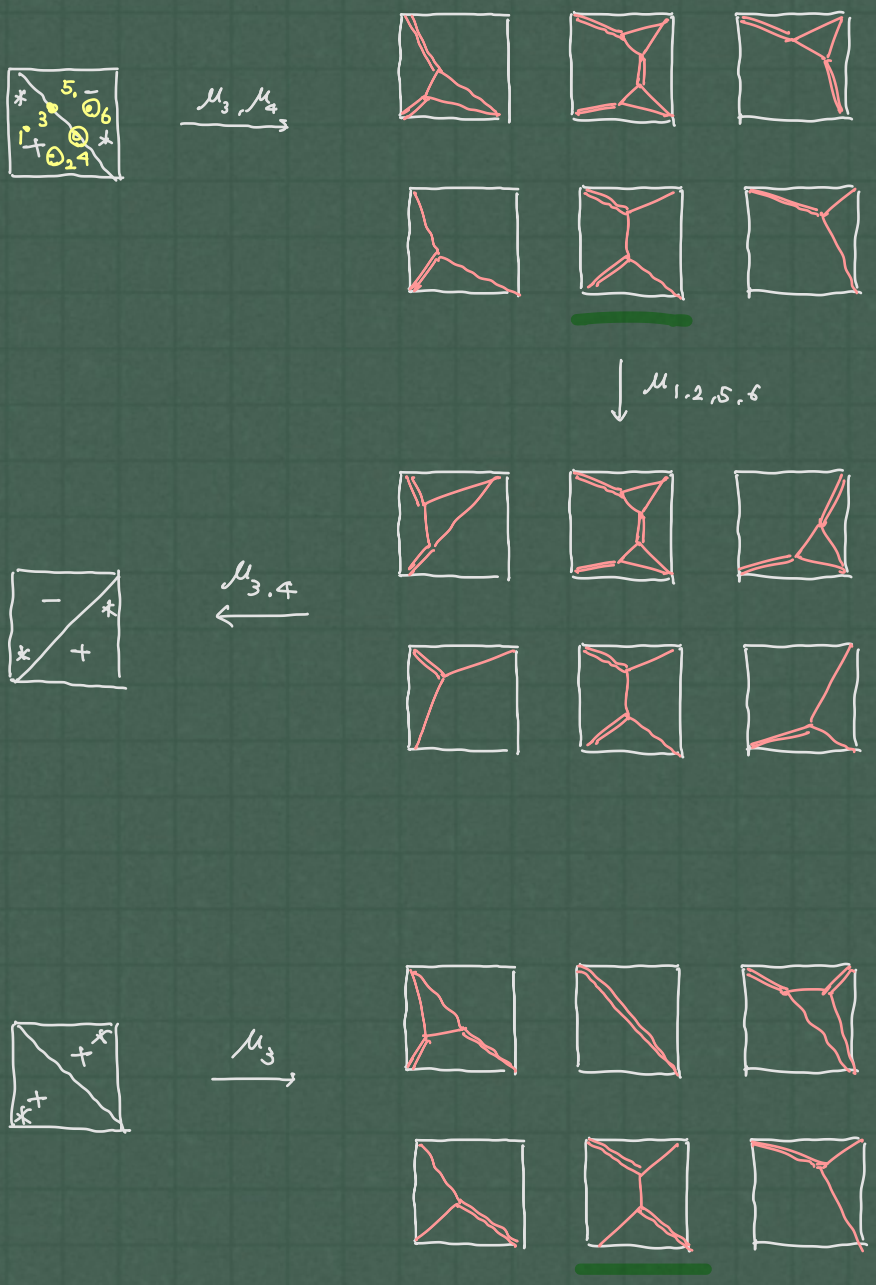
$e_1 e_1' =$

clusters of T



$\xleftrightarrow{\text{confirm compatibility}}$ $\mathscr{S}_{\mathfrak{sp}_4,T}[\partial^{-1}] = \mathscr{A}_{\mathfrak{sp}_4,T}$

⊘ $\Sigma = Q$: quadrilateral



$\mathrm{SimpWil}_{\Sigma}$ appears as a cluster variables of $\mathcal{A}_{\mathfrak{sp}_4,T}$ or $\mathcal{A}_{\mathfrak{sp}_4,Q}$ //