"Skein and cluster algebras of marked surfaces without punctures for $\mathfrak{sl}_3$"

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(joint work with Tsukasa Ishibashi)

[Ishibashi-Y.] arXiv: 2101.00643

$+ \begin{cases} \text{[Ishibashi-Y.] ——— of polygons} \\ \text{[Ishibashi-Y.] ——— for } \mathfrak{sp}_4 \\ \text{[Y.] Filtered \& graded } \mathfrak{sl}_3\text{-skein algebra} \end{cases}$

~ Quantum Geometry & Representation theory ~

March 04, 2021

Σ : a surface , $G = SL_2(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_2$

$$\begin{tikzcd} & \boxed{\mathrm{Hom}(\pi_1(\Sigma), G)/\!/G} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \boxed{\mathscr{S}^q_{\mathfrak{g},\Sigma}} \arrow[rr, leftrightarrow, "\text{[BW]}", "\text{[Muller]}"'] & & \boxed{\mathcal{A}_{\mathfrak{g},\Sigma}, \mathcal{X}_{\mathfrak{g},\Sigma}} \end{tikzcd}$$

- signed skein rel.
- skein rel. at markd pt

"naturally quantized"

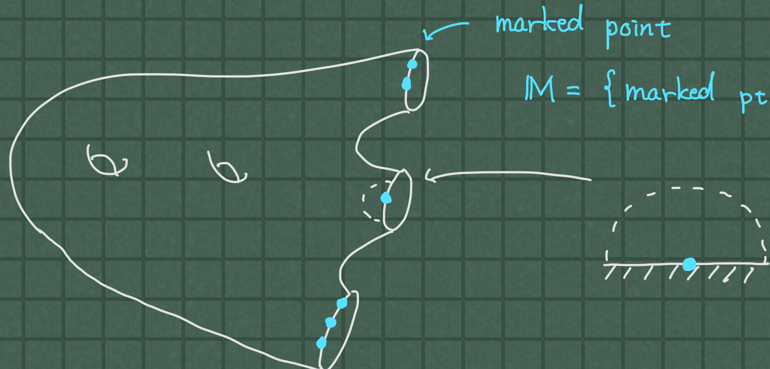
Today : • $\mathcal{A}^q_{\mathfrak{sl}_3,\Sigma} \subset \mathcal{S}^q_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \subset \mathcal{U}_{\mathcal{S}_q(\mathfrak{sl}_3,\Sigma)}$

• quantum positivity for "elevation-preserving web"

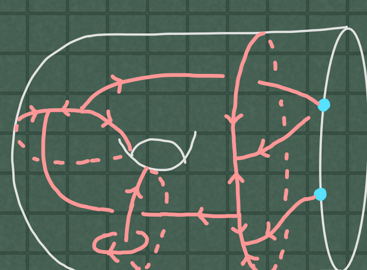
$$\begin{array}{ccccc} \mathcal{S}^q_{\mathfrak{sl}_3,\Sigma} & \xrightarrow[\text{Laurent expansion}]{} & \mathcal{U}_{\mathcal{S}_q(\mathfrak{sl}_3,\Sigma)} & \subset & \mathrm{Frac}\,\mathcal{S}^q_{\mathfrak{sl}_3,\Sigma} \\ \cup & & & & \\ \mathfrak{sl}_3\text{-web} & & & & \end{array}$$

with coefficients in $\mathbb{Z}_+[q^{\pm 1}]$

§ the $\mathfrak{sl}_3$-skein algebra

$\Sigma =$ 

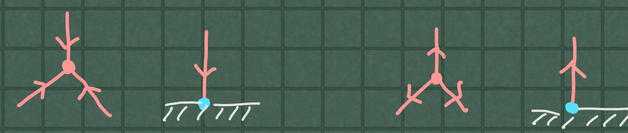
$\mathcal{S}^A_{\mathfrak{sl}_3,\Sigma}$: the $\mathfrak{sl}_3$-skein algebra of Σ consisting of $\mathfrak{sl}_3$-webs

e.g. $\mathfrak{sl}_3$-web 

◎ a tangled trivalent graph on Σ

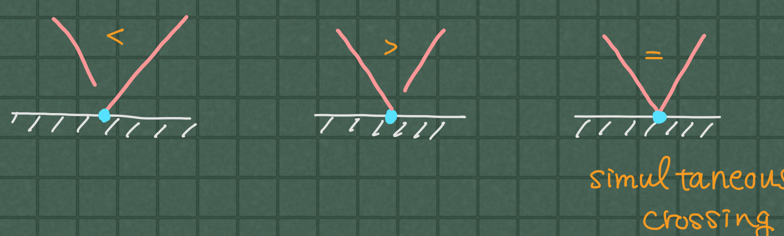
:$\Leftrightarrow$ an oriented uni-trivalent graph with

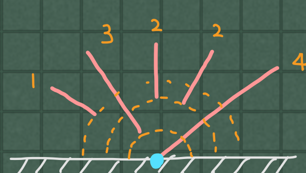
- sink and source vertices



- internal crossings on edges

- elevation at a marked point





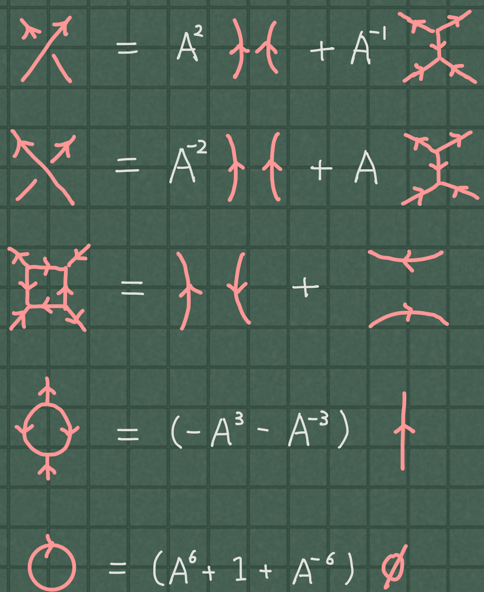
Definition (the $\mathfrak{sl}_3$-skein algebra of Σ)

$$\mathcal{S}^A_{\mathfrak{sl}_3,\Sigma} := \mathrm{span}_{\mathbb{Z}_A}\left\{ \begin{array}{l} \text{tangled trivalent} \\ \text{graphs on } \Sigma \end{array} \right\} \Big/ \begin{array}{l} (1)\ \mathfrak{sl}_3\text{-skein relations} \\ (2)\ \text{boundary } \mathfrak{sl}_3\text{-skein relations} \\ (3)\ \text{isotopy of } \Sigma \text{ rel. to } \partial\Sigma \end{array}$$

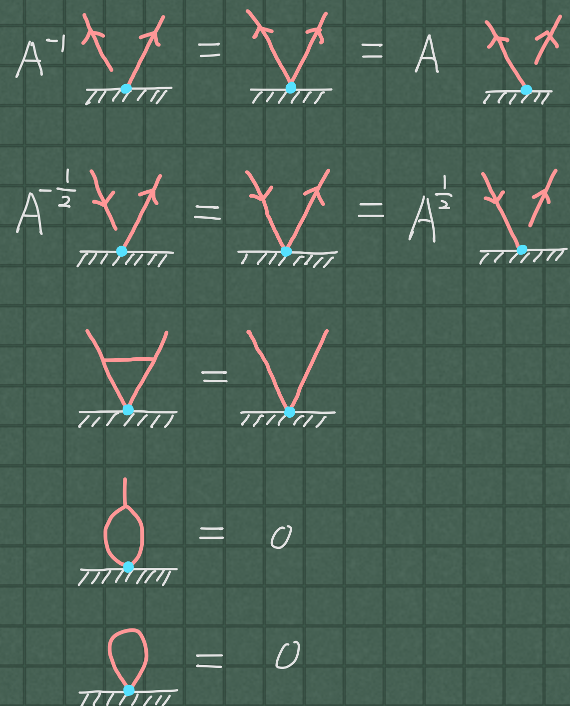
$\mathbb{Z}[A^{\pm\frac{1}{2}}]$

↑ $\mathfrak{sl}_3$-webs

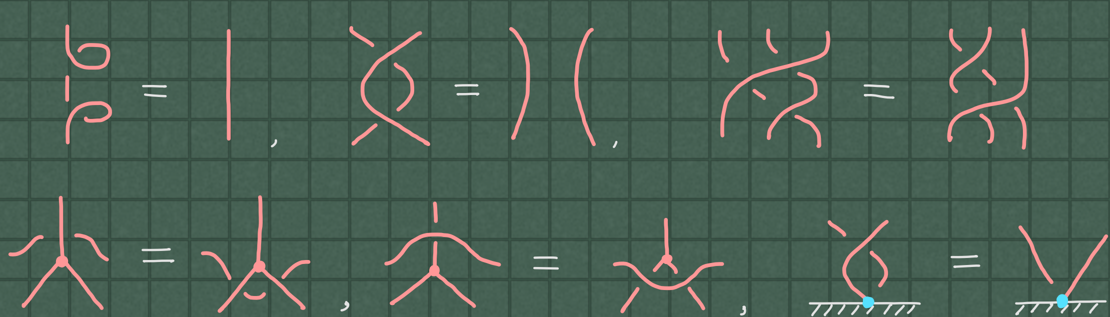
(1) $\mathfrak{sl}_3$ - skein relations [Kuperberg '96]



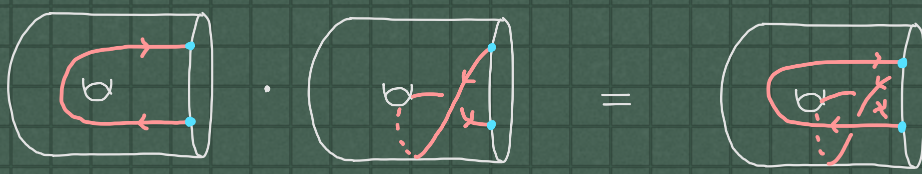
(2) boundary $\mathfrak{sl}_3$ - skein relations
[Ishibashi - Y '21]



$\rightsquigarrow$ realize Reidemeister moves



- multiplication of $\mathcal{S}^{A}_{\mathfrak{sl}_3,\Sigma}$



over under

Q. What is a basis of $\mathcal{S}^{A}_{\mathfrak{sl}_3,\Sigma}$

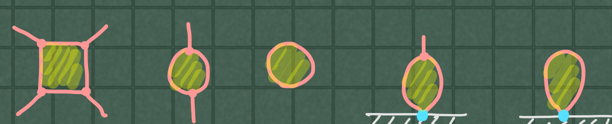
an $\mathfrak{sl}_3$-web $\longrightarrow$ apply skein relations

- remove over/under crossings
- remove n-gons $(n \leq 4)$

$\longrightarrow$ a sum of flat trivalent graphs with no elliptic faces

e.g. non-elliptic flat trivalent graph

elliptic faces



◎ basis webs BWeb_Σ

$\mathrm{BWeb}_\Sigma \ni$ $\mathfrak{sl}_3$ - webs represented by non-elliptic flat trivalent graphs

Theorem [IY] (the confluence theory in [Sikora-Westbury])

BWeb_Σ is a $\mathbb{Z}_A$ - basis of $\mathscr{S}^A_{\mathfrak{sl}_3,\Sigma}$

Generators of $\mathscr{S}^A_{\mathfrak{sl}_3,\Sigma}$ as a $\mathbb{Z}_A$- algebra

◎ elementary webs $\mathrm{EWeb}_\Sigma \subset \mathrm{BWeb}$

$\mathrm{EWeb}_\Sigma \ni G \;:\overset{}{\underset{\text{def}}{\Longleftrightarrow}}\; \nexists\, G_1, G_2$: basis webs s.t. $G = A^{\circledast} G_1 G_2$

◎ web cluster $\mathscr{C}$

an A - commutative subset of EWeb

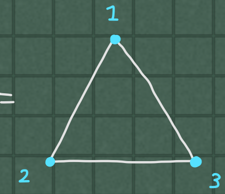
s.t. $\#\mathscr{C} = \#$ vertices of a $\mathfrak{sl}_3$ - triangulation

$\underset{?}{\Longleftrightarrow}$ a maximal A - commutative subset in EWeb

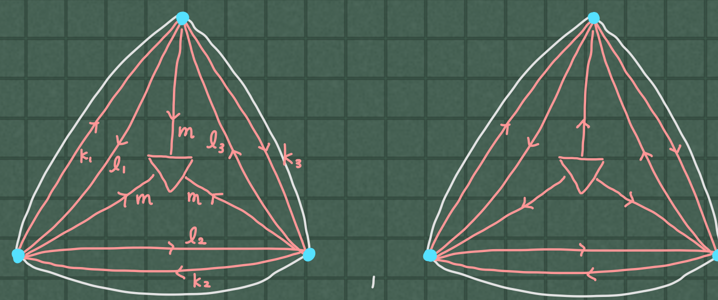
※ $\mathscr{C} \subset \mathrm{EWeb}_\Sigma \subset \mathrm{BWeb}_\Sigma \subset \mathscr{S}^A_{\mathfrak{sl}_3,\Sigma}$

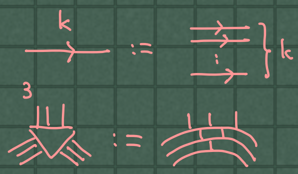
$\mathscr{C} \in \mathrm{CWeb}_\Sigma$

§ the $\mathfrak{sl}_3$-skein algebra of a triangle

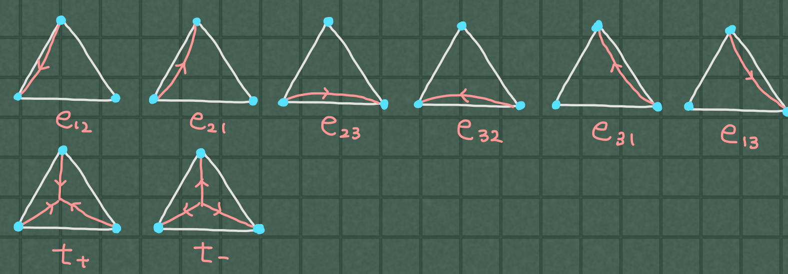
$T =$ 

Proposition [Kuperberg '96, D.Kim '07, Frohman-Sikora '20]

$\mathrm{BWeb}_T = \Big\{$  $\Big\}$



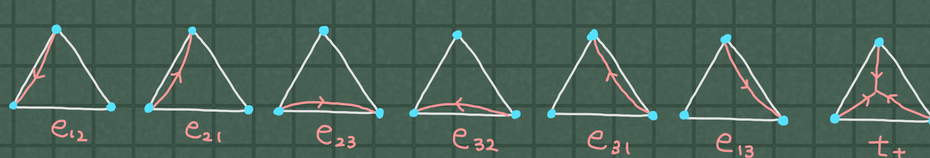
Proposition [IY]

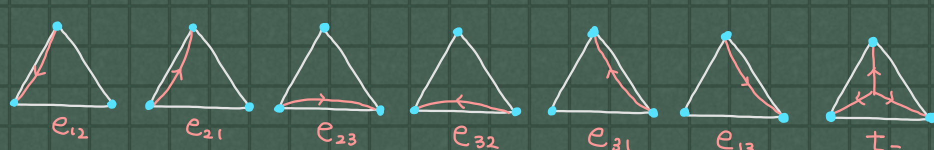
$\mathrm{EWeb}_T = \Big\{$  $\Big\}$

& it generates $\mathscr{S}_{\mathfrak{sl}_3, T}$.

the set of web clusters

Proposition [IY] $\quad \mathrm{CWeb}_T = \{\mathcal{C}_{(T,+)}, \mathcal{C}_{(T,-)}\}$

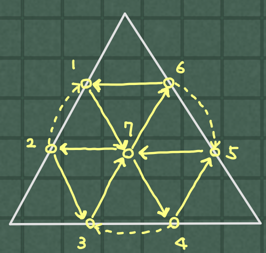
$\mathcal{C}_{(T,+)} = \Big\{$  $\Big\}$

$\mathcal{C}_{(T,-)} = \Big\{$  $\Big\}$

§ A quantum cluster algebra $\mathcal{A}^q_{\mathfrak{sl}_3,T}$ in $\mathcal{S}^q_{\mathfrak{sl}_3,T}$

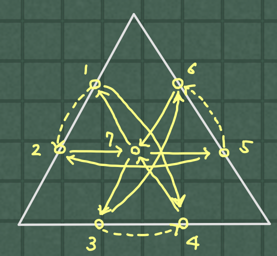
quivers

$I = \{1, 2, \dots, 6, 7\}$ $\quad I_f = \{1, 2, \dots, 6\}$ $\quad I_{uf} = \{7\}$



Q_+

$$\overset{\mu_7}{\longleftrightarrow}$$



Q_-

$$b_{ij} = \#\{ \overset{j}{\circ} \longrightarrow \overset{i}{\circ} \} + \frac{1}{2} \# \{ \overset{j}{\circ} \dashrightarrow \overset{i}{\circ} \}$$

skew symmetric,
$b_{ij} \in \frac{1}{2}\mathbb{Z}$ if $(i,j) \in I_f \times I_f$
$b_{ij} \in \mathbb{Z}$ otherwise

Seeds

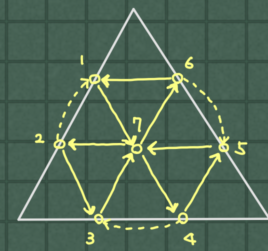
$$\left\{ \begin{array}{ll} B = (b_{ij})_{i,j \in I} & : \text{exchange matrix} \\ \mathbb{A} = (A_i)_{i \in I} & : \text{cluster } \mathcal{A}\text{-variables} \end{array} \right. \quad \overset{\mu_7}{\longleftrightarrow} \quad \left\{ \begin{array}{l} B' = (b'_{ij})_{i,j \in I} \\ \mathbb{A}' = (A'_i)_{i \in I} \end{array} \right.$$

alg. independent in a field $\mathcal{F}$

A_i $(i \in I_f)$: a frozen variables

◎ Exchange relation

$$\begin{cases} A_7 A_7' = A_1 A_3 A_5 + A_2 A_4 A_6 \\ A_i' = A_i \quad \text{if} \quad i \neq 7 \end{cases}$$



Q_T

T : triangle $\rightsquigarrow$ a seed pattern

$$S(\mathfrak{sl}_3, T) = \bullet \overset{\mu_7}{\text{———}} \bullet$$

$\rightsquigarrow$ the cluster algebra $\mathcal{A}_{\mathfrak{sl}_3, T}$

◎ the cluster algebra $\mathcal{A}_{\mathfrak{sl}_3, T}$ of T

:⟺ a subring of $\mathcal{F}$ generated by $\mathbb{A} \cup \mathbb{A}'$

and the inverses of frozen variables

In general

Σ : a marked surface $\rightsquigarrow$ a seed pattern $S(\mathfrak{sl}_3, \Sigma)$

$\rightsquigarrow$ the cluster algebra $\mathcal{A}_{S(\mathfrak{sl}_3, \Sigma)}$

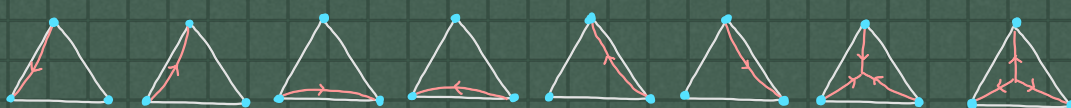
$\cup$

the surface subalgebra $\mathcal{A}_{\mathfrak{sl}_3, \Sigma}$

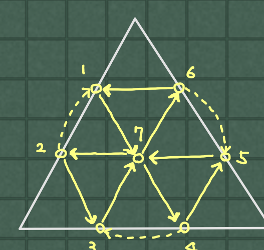
§ $\mathcal{A}^{q}_{\mathfrak{sl}_3,\Sigma} \subset \mathcal{S}^{q}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}]$

Define
$$\begin{array}{ccc} \mathcal{A}_{\mathfrak{sl}_3,T} & \longrightarrow & \mathcal{S}^{A=1}_{\mathfrak{sl}_3,T} \\ \cup & & \cup \\ A \cup A' & \longrightarrow & \mathrm{EWeb}_T = \mathcal{C}_{(T,+)} \cup \mathcal{C}_{(T,-)} \end{array}$$

by

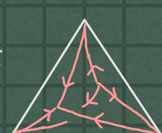


e_{12}	e_{21}	e_{23}	e_{32}	e_{31}	e_{13}	t_+	t_-
A_2	A_1	A_4	A_3	A_6	A_5	A_7	A_7'

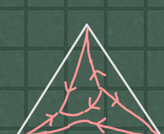
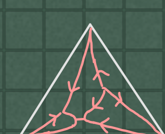


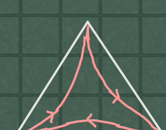
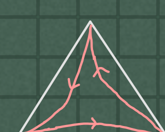
Q_+

Confirm the exchange relation

$t_+ t_-$ =  $= A^{-\frac{1}{2}}$ 

$A_7 A_7'$

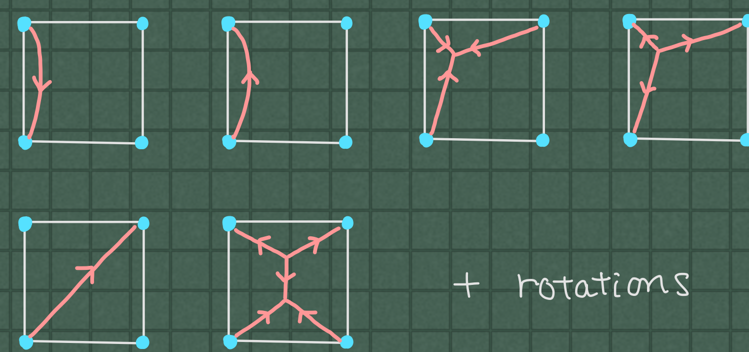
$= A^{\frac{3}{2}}$  $+ A^{-\frac{3}{2}}$ 

$= A^{\frac{3}{2}}$  $+ A^{-\frac{3}{2}}$ 

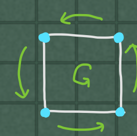
$[A_1 A_3 A_5]$ $\qquad$ $[A_2 A_4 A_6]$

- Q : a quadrilateral

$\mathrm{EWeb}_Q = \Big\{$



$+$ rotations



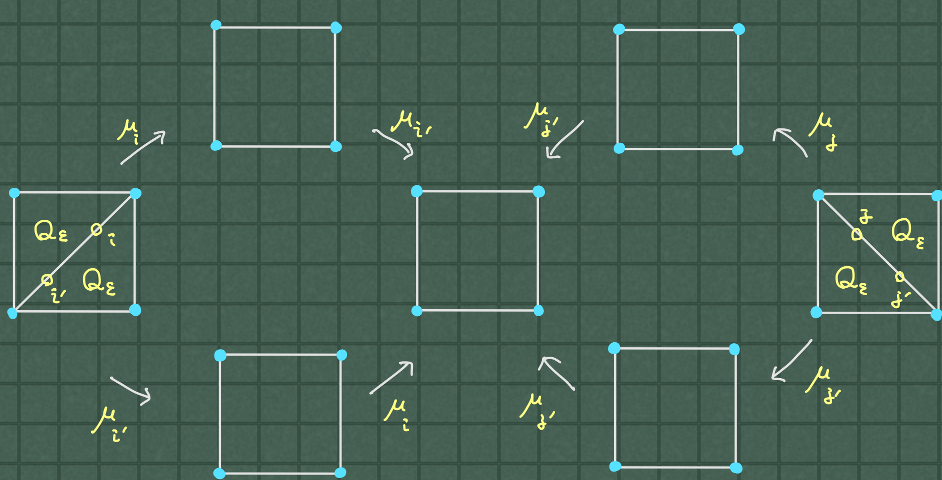
$\Big\}$

(in [IY])

$\mathrm{CWeb}_Q = \{ 50$ web clusters $\}$

(in [IY2])

Theorem [IY] $\quad \mathscr{A}^{q}_{\mathfrak{sl}_3,\Sigma} \subset \mathscr{S}^{q}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \overset{[Y]}{\subset} \mathrm{Frac}\,\mathscr{S}^{q}_{\mathfrak{sl}_3,\Sigma}$



seed pattern for the surface subalgebra

Theorem [IY 2] Σ : a disk with $\#\mathbb{M} = 3, 4, 5$

$$\mathcal{A}^{q}_{S_q(\mathfrak{sl}_3, \Sigma)} = \mathcal{S}^{q}_{\mathfrak{sl}_3, \Sigma}[\partial^{-1}] = \mathcal{U}_{S_q(\mathfrak{sl}_3, \Sigma)}$$

Theorem [IY 3] $\mathfrak{g} = \mathfrak{sp}_4$

$$\mathcal{A}^{q}_{\mathfrak{sp}_4, \Sigma} \subset \mathcal{S}^{q}_{\mathfrak{sp}_4, \Sigma}[\partial^{-1}]$$

§ Laurent expressions & quantum positivity

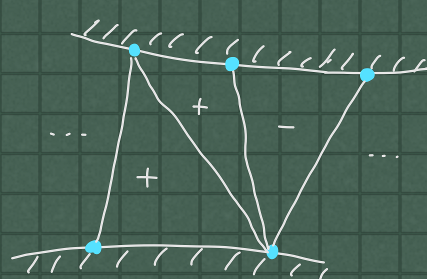
Δ : an ideal triangulation of Σ.

$t(\Delta)$: triangles in Δ

Definition

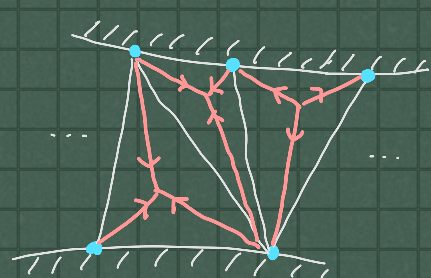
- a decorated triangulation $\boldsymbol{\Delta} = (\Delta, \mathbb{s})$ of Σ

$$:\underset{\mathrm{def}}{\Longleftrightarrow} \begin{cases} \Delta : \text{an ideal triangulation of } \Sigma \\ \mathbb{s} : t(\Delta) \longrightarrow \{+, -\}, \text{ a map} \end{cases}$$



- web cluster $\mathscr{C}_{\boldsymbol{\Delta}}$ associated with $\boldsymbol{\Delta}$

$$\mathscr{C}_{\boldsymbol{\Delta}} := \bigcup_{T \in t(\Delta)} \mathscr{C}_{(T, \mathbb{s}(T))}$$



Theorem [IY] $\forall\, G \in \mathrm{BWeb}$, $\exists\, J_G$: monomial in $\mathscr{C}_{\boldsymbol{\Delta}}$

s.t. $G \cdot J_G \in \langle \mathscr{C}_{\boldsymbol{\Delta}} \rangle_{\mathrm{alg}} \subset \mathscr{S}^{A}_{\mathfrak{sl}_3, \Sigma}$

Corollary

- Any $\mathfrak{sl}_3$-web has a Laurent expression in $\mathcal{C}_\Delta$

- $\mathcal{A}^q_{\mathfrak{sl}_3,\Sigma} \subset \mathcal{S}^q_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \subset \mathcal{U}_{\mathcal{S}_q(\mathfrak{sl}_3,\Sigma)}$

q-Laurent phenomenon

$\uparrow$ quantization

$\mathcal{O}(\mathcal{A}_{\mathcal{S}(\mathfrak{sl}_3,\Sigma)})$

$\cap$

$\mathcal{O}(\mathcal{A}_{SL_3,\Sigma})$

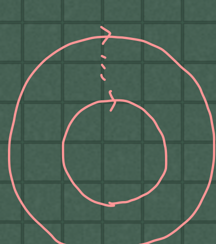
◎ "quantum positivity"

S : a certain subset of $\mathfrak{sl}_3$-webs

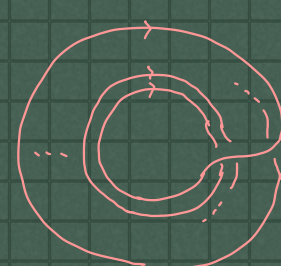
S has quantum positivity w.r.t. Δ

$\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ ${}^\forall G \in S$ has a Laurent expression in $\langle \mathcal{C}_\Delta \rangle_{\text{alg}}$ with coefficients in $\mathbb{Z}_+[q^{\pm 1}]$

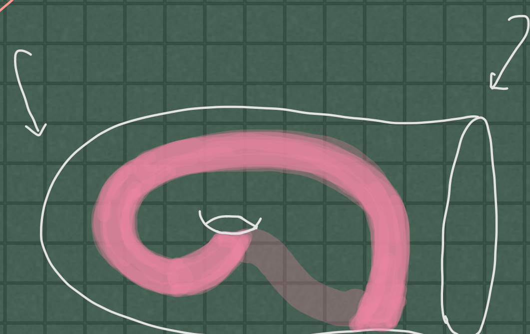
typical example of S



embeddings along simple loops.



$\text{bang}_\Sigma =$



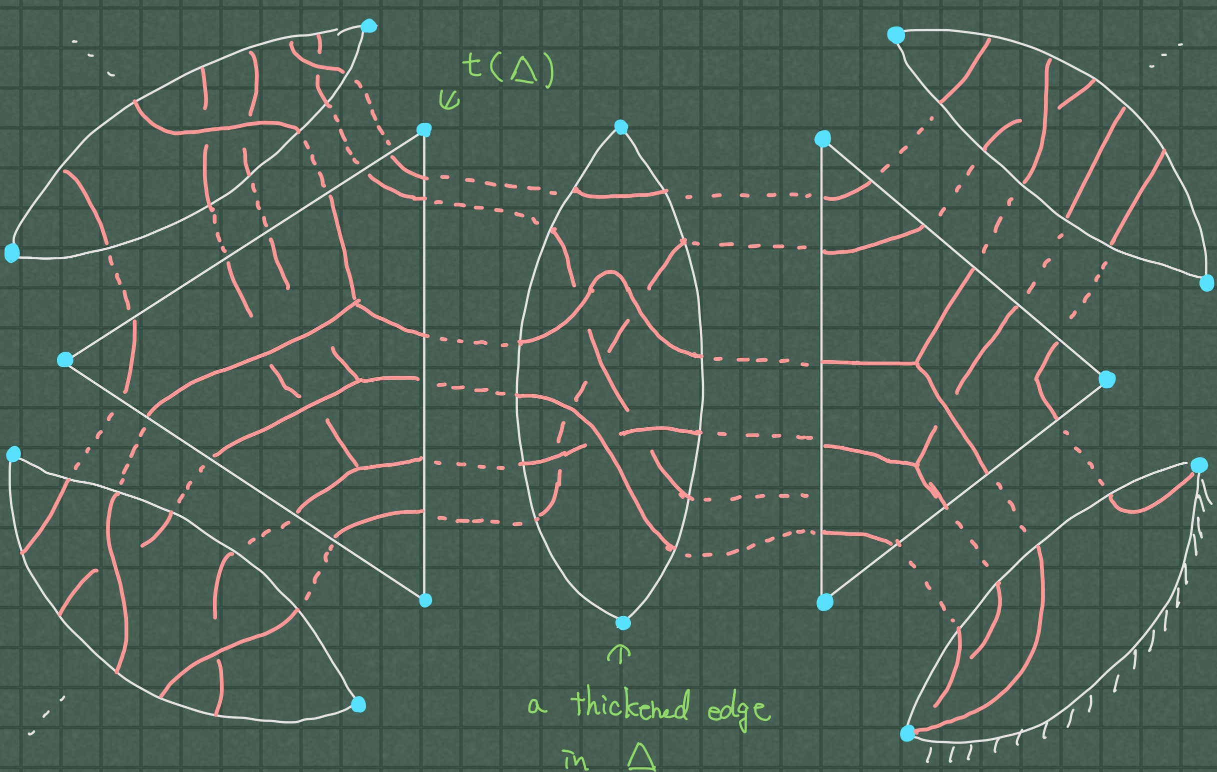
$= \text{brace}_\Sigma$

Theorem [IY] Δ : a decorated triangulation

$\mathrm{elev}_{\Delta} = \{$ elevation-preserving webs w.r.t. Δ $\}$

has positivity for $\mathcal{C}_{\Delta}$

◎ elevation-preserving web



Remark elev_{Δ} contains $\mathrm{bang}_{\Sigma} \cup \mathrm{brace}_{\Sigma}$ for any Δ

Theorem [IY3] Elevation preserving $\mathfrak{sp}_4$-webs have "quantum positivity".

~ Thank you for listening ~