

色付きジョーンズ多項式の tail と q -級数

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Friday Tea Time Zoom Seminar, (November 13, 2020)

- ① tail に関するこれまでの主な結果
- ② 結び目の量子不変量と線形スケイン理論
- ③ 線形スケイン理論による Braiding の分解公式
- ④ 結び目の tail から q -級数の恒等式を得る処方箋

色付きジョーンズ多項式

▶ 色付きジョーンズ多項式とは

Lie 代数 $\mathfrak{sl}(2, \mathbb{C})$ の $(n+1)$ -次元の既約表現 V_{n+1} に対して定義される結び目, 絡み目の "多項式" に値を取る不変量の族 $\{J_K^{\mathfrak{sl}_2}(n)\}_n$. ($J_{K,n}^{\mathfrak{sl}_2}(q)$ と書いたりもする.)

■ Remark

- 色付きジョーンズ多項式 $\{J_K^{\mathfrak{sl}_2}(n)\}_n$ は $q^{\frac{1}{2}} \mathbb{Z}[q^{\pm 1}]$ に値を取る.
- V_2 から得られる色付きジョーンズ多項式 $J_K^{\mathfrak{sl}_2}(1)$ がジョーンズ多項式.

色付きジョーンズ多項式 $J_K^{\mathfrak{sl}_2}(n)$ に $\pm q^{\frac{1}{2}}$ をかけて q -多項式に正規化した不変量

$$\hat{J}_K^{\mathfrak{sl}_2}(n) = a_0 + a_1 q + a_2 q^2 + \cdots, \quad (a_0 > 0)$$

を考える事が出来る.

▶ 色付きジョーンズ多項式の \mathfrak{sl}_2 -tail とは

極限 " $\lim_{n \rightarrow \infty} \hat{J}_K^{\mathfrak{sl}_2}(n)$ " として得られる形式的冪級数 $\mathcal{T}_K^{\mathfrak{sl}_2}(q) \in \mathbb{Z}[[q]]$.

色付きジョーンズ多項式の係数の安定性

■ **Theorem ([Armond, 2013])** L を adequate link とする. このとき, 形式的 q -級数 $\mathcal{T}_L^{sl_2}(q) \in \mathbb{Z}[[q]]$ が存在して, $\mathcal{T}_L^{sl_2}(q) - \hat{\mathcal{J}}_{L,n}^{sl_2}(q) \in q^{n+1}\mathbb{Z}[[q]]$.

■ **Theorem ([Garoufalidis-Lê, 2015])** L を交代絡み目とする. このとき, 任意の非負整数 k に対して $\{\hat{\mathcal{J}}_{K,n}^{sl_2}(q)\}_n$ は k -stable. (0 -stable \iff tail の存在)

■ **Example 8** の字結び目の色付きジョーンズ多項式の係数の表

$n=2$	1	-1	-1	0	2	0	-2	0	3	0	-3	0	3	0	-3	0
$n=3$	1	-1	-1	0	0	3	-1	-1	-1	-1	5	-1	-2	-2	-1	6
$n=4$	1	-1	-1	0	0	1	2	0	-2	-1	-1	1	3	1	-2	-3
$n=\infty$	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0	-1
$n=2$	0	0	2	-1	-2	-1	3	0	-3	0	4	0	-2	1		
$n=3$	0	0	2	-1	-2	-1	-1	5	-1	-1	-2	-1	7			
$n=4$	0	0	2	-1	-2	-1	-1	1	4	1	-2	-2				
$n=\infty$	0	0	2	-1	-2	-1	-1	1

引き算 & 左に n シフト

▶ 色付き \mathfrak{g} ジョーンズ多項式 (結び目の量子 $(\mathfrak{g}, V_\lambda)$ 不変量)

単純 Lie 代数 \mathfrak{g} の最高ウェイト λ を持つ既約表現 V_λ に対して定義される結び目 (絡み目) の不変量の族 $\{J_K^{\mathfrak{g}}(\lambda)\}_{\lambda \in \Lambda}$.

■ Example $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ とすると, $(k, l) \in \Lambda = \mathbb{N} \times \mathbb{N}$ に対して $J_K^{\mathfrak{sl}_3}(k, l)$ が定義される. 特に, $\{J_K^{\mathfrak{sl}_3}(k, 0)\}$ を **一行色付き \mathfrak{sl}_3 ジョーンズ多項式** と呼ぶ.

■ Theorem (Integrality theorem [Lê, 2000])

$$J_K^{\mathfrak{g}}(\lambda) \in q^{\frac{1}{2}} \mathbb{Z}[q^{\pm 1}]$$

■ Remark

- $\mathfrak{g} = \mathfrak{sl}_2$ 以外での”非自明”な結び目に対する色付きジョーンズ多項式の明示式はほとんど知られていない.
- q -多項式への正規化 $\{\hat{J}_K^{\mathfrak{g}}(\lambda)\}_{\lambda \in \Lambda}$ を考えることは出来るが, 一般にその”極限”である \mathfrak{g} -tail が存在するかは知られていない.

■ Theorem ([Garoufalidis-Vuong, 2017]) K をトーラス結び目, \mathfrak{g} をランク 2 の単純 Lie 代数とする. このとき, 任意の基本表現 λ に対して $\{\hat{J}_K^{\mathfrak{g}}(n\lambda)\}_{n \in \mathbb{N}}$ は k -stable. ($k \in \mathbb{N}$)

tail に関して得られた結果の紹介

- **Theorem ([Y. 2018])** Anti-parallel な $(2, 2m)$ -torus link $T_{\Leftarrow}(2, 2m)$ に対して, $J_K^{\mathfrak{sl}_3}(k, 0)$ に関する二通りの明示式を与え, その \mathfrak{sl}_3 -tail から以下の恒等式を得た.

$$\sum_{i=0}^{\infty} q^{-2i} q^{m(i^2+2i)} \frac{(1-q^{i+1})^3(1+q^{i+1})}{(1-q)^2(1-q^2)}$$

$$= \frac{(q)_{\infty}}{(1-q)(1-q^2)} \sum_{k_1 \geq k_2 \geq \dots \geq k_m \geq 0} \frac{q^{-2k_m} q^{\sum_{i=1}^m k_i^2 + 2k_i}}{(q)_{k_1-k_2} (q)_{k_2-k_3} \cdots (q)_{k_{m-1}-k_m} (q)_{k_m}^2}$$

- **Remark** 左辺は [Bringmann-Kaszian-Milas, 2019] の \mathfrak{sl}_3 false theta series の "diagonal summand" と一致する.

- **Theorem ([Y. 2020])** Parallel な $(2, 2m)$ -torus link $T_{\rightarrow}(2, 2m)$ のに対して, \mathfrak{sl}_2 -tail と \mathfrak{sl}_3 -tail は有理関数の差を除いて一致する.

- **Theorem ([Y. 2020])** "minus-adequate" な向き付き絡み目 K に対して, 一行色付きジョーンズ多項式の極限 $\mathcal{T}_K^{\mathfrak{sl}_3}(q)$ が存在する.

結び目図式による結び目の定義

▶ 絡み目とは

|| \mathbb{R}^3 に埋め込まれた $\sqcup^l S^1$ の全域 isotopy 類.

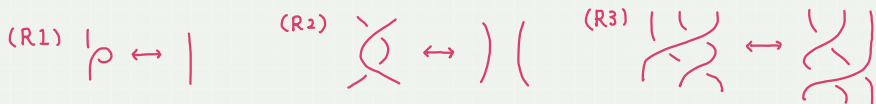
▶ 絡み目図式とは

|| \mathbb{R}^3 に埋め込まれた $\sqcup^l S^1$ の \mathbb{R}^2 への generic な射影に交点における上下情報を付加した図.

■ **Remark** l を絡み目の成分数といい, 特に $l=1$ のときに結び目と呼ぶ. S^1 をアニュラス $S^1 \times [0, 1]$ に置き換えたものを **枠付き絡み目**と呼ぶ.

▶ 絡み目図式と絡み目の対応

|| 絡み目図式 D_1 と D_2 が **Reidemeister move** の列と \mathbb{R}^2 の isotopy で移り合うなら, D_1 を絡み目図式にもつ絡み目と D_2 を絡み目図式にもつ絡み目は一致する.



↓ framed : (R1)を (R0)
に replace



枠付き結び目の量子不変量

▶ 結び目の量子不変量 (色付きジョーンズ多項式) の定義

- ① 表現圏を用いた定義,
- ② 結び目図式を用いた定義,
- ③ その他いろいろ

▶ 表現圏を用いた量子不変量の構成 (概要)

- strict なリボン圏 \mathcal{C} を固定すると, \mathcal{C} の対象で色付けしたタングル図式の圏から \mathcal{C} への関手 $F_{\mathcal{C}}$ が一意に存在する.
- $c \in \mathcal{C}$ で色付けられた絡み目 L_c は単位対象から単位対象への射なので, 不変量 $F_{\mathcal{C}}(L_c) \in \text{End}(1)$ が得られる.
- 特に, \mathcal{C} を量子群 $U_q(\mathfrak{g})$ の有限次元表現の圏とすると $F_{\mathcal{C}}(L_c)$ として色付き \mathfrak{g} ジョーンズ多項式が得られる.

▶ 線形スケイン理論を用いた量子不変量の構成 (概要)

- 結び目 L の結び目図式 D_L を描き, "Jones-Wenzl 射影子" を乗せる.
- "スケイン関係式" によって交点を持たない図式の線形和 $D_L = \sum_i D_i$ に分解する.
- 各 D_i の "elliptic face" をスケイン関係式で消していくと空の図式 \emptyset となる.
- 以上の操作で $D_L = f_L(q)\emptyset$ となり, 得られた多項式 $f_L(q)$ が色付き \mathfrak{g} ジョーンズ多項式が得られる.

表現圏を用いた量子不変量の構成

a (k, ℓ) -tangle diagram

$\stackrel{\text{def}}{\Leftrightarrow}$ a generic immersion of arcs & loops into $\mathbb{R} \times [0, 1]$

s.t. • intersection point = \times

$$\bullet \partial\{\text{arcs}\} = \{(1,0), (2,0), \dots, (k,0)\} \cup \{(1,1), (2,1), \dots, (\ell,1)\}$$



$(5, 3)$ -tangle diagram

the set of framed (k, ℓ) -tangles

$$:= \left\{ (k, \ell)\text{-tangle diagram} \right\} / \begin{matrix} (\Omega 0), (\Omega 2), \\ (\Omega 3), \text{isotopy} \end{matrix}$$

Fix a strict ribbon category \mathcal{C}

$$\left[\begin{array}{l} \text{braiding } C_{V,W}: V \otimes W \rightarrow W \otimes V \\ \text{duality } \left\{ \begin{array}{l} * : V \rightarrow V^* \\ b_V: \mathbf{1} \rightarrow V \otimes V^* \\ d_V: V^* \otimes V \rightarrow \mathbf{1} \end{array} \right. \\ \text{twist } \theta_V: V \rightarrow V, \text{ isom.} \end{array} \right]$$

a category of \mathcal{C} -colored framed tangles $\mathcal{T}_{\mathcal{C}}$

$\stackrel{\text{def}}{\Leftrightarrow}$ $\text{Obj } \mathcal{T}_{\mathcal{C}}$: a finite sequence of $\text{Obj } \mathcal{C} \times \{+, -\}$
 $\hookrightarrow \emptyset$: the empty sequence.

$$\text{Obj } \mathcal{T}_{\mathcal{C}} \ni \eta_1 = ((V_1, \varepsilon_1), (V_2, \varepsilon_2), \dots, (V_k, \varepsilon_k))$$

$$\eta_2 = ((W_1, \delta_1), (W_2, \delta_2), \dots, (W_\ell, \delta_\ell))$$

$\text{Mor}(\eta_1, \eta_2)$: an oriented framed (k, ℓ) -tangle

s.t. • each component is colored by $\text{Obj } \mathcal{C}$

$$\bullet \begin{array}{cc} \frac{(v, 1)}{\downarrow V_i} \text{ if } \varepsilon_i = +, & \frac{(v, 1)}{\uparrow V_i} \text{ if } \varepsilon_i = - \\ \frac{w_i}{\downarrow} \text{ if } \delta_i = +, & \frac{w_i}{\uparrow} \text{ if } \delta_i = - \end{array}$$

a category of \mathcal{L} -colored framed tangles $\mathcal{T}_{\mathcal{L}}$

- $\text{Obj } \mathcal{T}_{\mathcal{L}}$: a finite sequence of $\text{Obj } \mathcal{L} \times \{+, -\}$
 $\cup \emptyset$: the empty sequence.

$$\text{Obj } \mathcal{T}_{\mathcal{L}} \ni \eta_1 = ((V_1, \varepsilon_1), (V_2, \varepsilon_2), \dots, (V_k, \varepsilon_k))$$

$$\eta_2 = ((W_1, \delta_1), (W_2, \delta_2), \dots, (W_\ell, \delta_\ell))$$

- $\text{Mor}(\eta_1, \eta_2)$: an oriented framed (k, ℓ) -tangle

s.t. • each component is colored by $\text{Obj } \mathcal{L}$

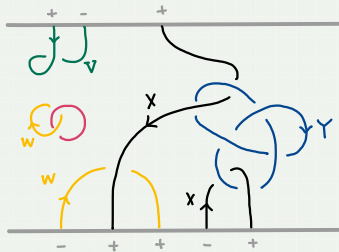
$$\begin{array}{ll} \cdot \begin{array}{c} \overset{(i, +)}{\downarrow} \\ \downarrow_{V_i} \end{array} \text{ if } \varepsilon_i = +, & \begin{array}{c} \overset{(i, -)}{\downarrow} \\ \downarrow_{V_i} \end{array} \text{ if } \varepsilon_i = - \\ \begin{array}{c} \downarrow_{W_j} \\ \delta_j = + \end{array} & \begin{array}{c} \downarrow_{W_j} \\ \delta_j = - \end{array} \end{array}$$

- composition

$$\boxed{T_1} \circ \boxed{T_2} := \boxed{\begin{array}{c} T_1 \\ \hline T_2 \end{array}}$$

- tensor product

$$\boxed{T_1} \otimes \boxed{T_2} := \boxed{\begin{array}{c|c} T_1 & T_2 \end{array}}$$



\mathcal{L} -colored $(5, 3)$ -tangle diagram

Theorem [Reshetikhin-Turaev, 1990]

$\exists!$ $F_e: \mathcal{T}_e \rightarrow \mathcal{C}$, a \otimes -preserving functor
 s.t. $F((v,+)) = V$, $F((v,-)) = V^*$

$$\begin{array}{c} \text{w} \\ \swarrow \searrow \\ \text{v} \end{array} = C_{v,w} \quad \begin{array}{c} \text{w} \\ \swarrow \nearrow \\ \text{v} \end{array} = C_{w,v}^{-1} \quad \begin{array}{c} \text{w} \\ \nearrow \searrow \\ \text{v} \end{array} = C_{w,v^*}^{-1} \quad \begin{array}{c} \text{w} \\ \nearrow \nearrow \\ \text{v} \end{array} = C_{v^*,w}$$

$$\begin{array}{c} \text{v} \\ \swarrow \searrow \\ \text{w} \end{array} = C_{w^*,v}^{-1} \quad \begin{array}{c} \text{v} \\ \swarrow \nearrow \\ \text{w} \end{array} = C_{v,w^*} \quad \begin{array}{c} \text{v} \\ \nearrow \searrow \\ \text{w} \end{array} = C_{v,w^*} \quad \begin{array}{c} \text{v} \\ \nearrow \nearrow \\ \text{w} \end{array} = C_{w^*,v}^{-1}$$

$$\text{v} \downarrow = \text{id}_V \quad \text{v} \uparrow = \text{id}_{V^*} \quad \text{v} \downarrow \circlearrowleft = \theta_V \quad \text{v} \uparrow \circlearrowright = \theta_{V^*}^{-1}$$

$$\text{v} \curvearrowright = d_V \quad \text{v} \curvearrowleft = b_V$$

$\rightsquigarrow L$: an \mathcal{C} -colored framed link.

$$\Rightarrow F_e(L) \in \text{End}(\mathbb{1})$$

e.g. (the colored \mathfrak{g} Jones polynomial)

\mathfrak{g} : a simple Lie algebra

$\text{Rep}_f U_q(\mathfrak{g})$: the category of finite dimensional
 \mathcal{C} representations of the quantum
 group $U_q(\mathfrak{g})$. ($q = "q^2"$: a formal variable)

$L = L_1 \cup L_2 \cup \dots \cup L_\ell$: a framed link

$V_i \in \text{Rep}_f U_q(\mathfrak{g})$ is a coloring of L_i

the $(\mathfrak{g}, (V_1, \dots, V_\ell))$ -colored Jones polynomial

$J_L^{\mathfrak{g}}(V_1, V_2, \dots, V_\ell) \in \mathbb{C}(q^{\pm 1/2})$ is defined by

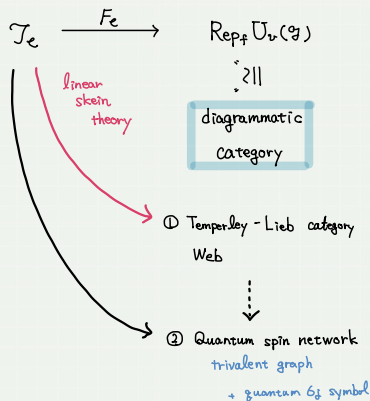
$$F_e(L)(1) = J_L^{\mathfrak{g}}(V_1, V_2, \dots, V_\ell) \cdot 1$$

(\times : D is determined by \mathfrak{g})

線形スケイン理論を用いた量子不変量の構成

$$\mathcal{L} = \text{Rep}_F U_{\mathfrak{g}}(\mathfrak{g})$$

$\mathcal{T}_{\mathcal{L}} = \mathcal{L}$ -colored framed tangles



"Linear skein theory"

= a functor from $\mathcal{T}_{\mathcal{L}}$ to
 a diagrammatic representation
 of $\text{Fun Rep}_F U_{\mathfrak{g}}(\mathfrak{g})$
 or $\text{Kar}(\text{Fun Rep}_F U_{\mathfrak{g}}(\mathfrak{g}))$

the Kauffman bracket ($\mathfrak{g} = \mathfrak{sl}_2$)

$$| = | 1 = \downarrow \begin{matrix} V_2 \otimes V_2^* \\ \text{the 2-dim.} \\ \text{irreducible rep.} \end{matrix}$$

$$|^n = \underbrace{|| \dots ||}_n = | V_2 \otimes \dots \otimes V_2$$

the Kauffman bracket ($\mathcal{G} = \mathcal{sl}_2$)

skein relation $\left\{ \begin{array}{l} \diagdown = q^{\frac{1}{2}} \left(+ q^{-\frac{1}{2}} \diagup \right) \\ \bigcirc = -[2] \emptyset = (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \emptyset \end{array} \right.$

• construction of the color $|V_{n+1}$
 = the Jones - Wenzl projector \downarrow^n

i.e. $\downarrow^n = \frac{\downarrow^{n-1}}{\downarrow^{n-1}} : V_2^{\otimes n} \rightarrow \text{Sym}^n V_2 \hookrightarrow V_2^{\otimes n}$

Diagrammatic definition $\left([n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \right)$

$V_{n+1} : \downarrow^n = \downarrow^{n-1} | + \frac{[n-1]}{[n]} \downarrow^{n-1} \downarrow_1^1$

$\rightsquigarrow K = \bigcirc$

then, $\downarrow^n = J_{K,n}^{\mathcal{sl}_2}(\emptyset) \emptyset$

the A_2 bracket ($\mathcal{G} = \mathcal{sl}_3$) $\uparrow = \uparrow_{V_{0,0}} = \uparrow_{V_{0,1}}$

skein relation $\left\{ \begin{array}{l} \diagup \diagdown = q^{\frac{1}{3}} \left(- q^{-\frac{1}{3}} \diagdown \diagup \right) \\ \diagdown \diagup = q^{-\frac{1}{3}} \left(- q^{\frac{1}{3}} \diagup \diagdown \right) \\ \square = \diagdown \diagup + \diagup \diagdown \\ \uparrow \downarrow = [2] \uparrow, \quad \bigcirc = \bigcirc = [3] \emptyset \end{array} \right.$

$V_{(n,0)} : \downarrow^n = \downarrow^{n-1} \downarrow - \frac{[n-1]}{[n]} \downarrow^{n-1} \downarrow_1^1$

$V_{(m,n)} : \downarrow_m^m \downarrow_n^n = \sum_{k=0}^{\min\{m,n\}} (-1)^k \frac{[m][n]}{[m+k][n-k]} \downarrow_{m+k}^{m+k} \downarrow_{n-k}^{n-k}$

$\rightsquigarrow \downarrow_m^m \downarrow_n^n = J_{K,(m,n)}^{\mathcal{sl}_3}(\emptyset) \emptyset$

■ **Example** (時間があれば計算例)

ツイスト公式 ($\mathfrak{g} = \mathfrak{sl}_2$ の場合)

Theorem[Yamada, 1989]

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \sum_{k=0}^n q^{\frac{1}{4}(-n^2+2k)} \frac{\binom{n}{q}_n}{\binom{n}{q}_k \binom{n}{q}_{n-k}} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$$

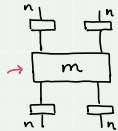
Diagram description: The left side shows two vertical strands, each with a box labeled 'n' at the top. The strands cross each other. The right side shows a sum over k from 0 to n. Each term consists of a coefficient $q^{\frac{1}{4}(-n^2+2k)}$, a fraction of q-binomial coefficients $\frac{\binom{n}{q}_n}{\binom{n}{q}_k \binom{n}{q}_{n-k}}$, and a diagram of two vertical strands with boxes labeled 'n' at the top and 'k' at the bottom. The strands are connected by two horizontal arcs, each labeled 'n-k'.

Theorem[Masbaum, 2003]

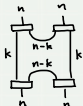
$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \sum_{k=0}^n (-1)^{n-k} q^{\frac{1}{2}(-n^2-n+2k^2+k)} \frac{\binom{n}{q}_n^2}{\binom{n}{q}_k^2 \binom{n}{q}_{n-k}} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$$

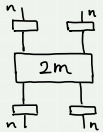
Diagram description: The left side shows two vertical strands, each with a box labeled 'n' at the top. The strands cross each other. The right side shows a sum over k from 0 to n. Each term consists of a coefficient $(-1)^{n-k} q^{\frac{1}{2}(-n^2-n+2k^2+k)}$, a fraction of q-binomial coefficients $\frac{\binom{n}{q}_n^2}{\binom{n}{q}_k^2 \binom{n}{q}_{n-k}}$, and a diagram of two vertical strands with boxes labeled 'n' at the top and 'k' at the bottom. The strands are connected by two horizontal arcs, each labeled 'n-k'.

Theorem[Y. 2017]

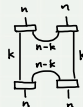
m


$$= (-1)^{n-km} q^{\frac{n-km}{2}} \sum_{n \geq k_1 \geq k_2 \geq \dots \geq k_m \geq 0} (-1)^{\sum_{i=1}^m k_i} q^{\frac{1}{2} \sum_{i=1}^m (k_i^2 + k_i)}$$

$$\times \frac{(q)_n}{(q)_{n-k_1} (q)_{k_1-k_2} \dots (q)_{k_{m-1}-k_m} (q)_{k_m}}$$




$$= q^{-\frac{m}{2}(n^2+2n)} \sum_{n \geq k_1 \geq k_2 \geq \dots \geq k_m \geq 0} (-1)^{n-km} q^{\frac{n-km}{2}} q^{\frac{m}{2} \sum_{i=1}^m (k_i^2 + k_i)}$$

$$\times \frac{(q)_n^2}{(q)_{n-k_1} (q)_{k_1-k_2} \dots (q)_{k_{m-1}-k_m} (q)_{k_m}^2}$$


ツイスト公式 ($g = sl_3, \lambda = (n, 0)$ の場合)**Theorem[Y. 2017]** (anti-parallel case)

$$\begin{aligned}
 & \text{Diagram: A box labeled } 2m \text{ with } n \text{ strands entering from the top and } n \text{ strands exiting from the bottom. The top and bottom strands are connected by } m \text{ crossings.} \\
 & = q^{-\frac{2m}{3}(n^2+3n)} \sum_{n \geq k_1 \geq k_2 \geq \dots \geq k_m \geq 0} q^{n-k_m} q^{\sum_{i=1}^m (k_i^2 + 2k_i)} \\
 & \quad \times \frac{(q)_n^2}{(q)_{n-k_1} (q)_{k_1-k_2} \dots (q)_{k_{m-1}-k_m} (q)_{k_m}^2} \text{Diagram: A braiding diagram with } n \text{ strands, } m \text{ crossings, and } k_i \text{ strands between crossings.}
 \end{aligned}$$

Theorem[Y. 2020]

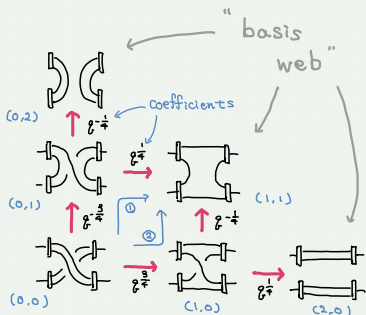
$$\begin{aligned}
 & \text{Diagram: A box labeled } 2m \text{ with } n \text{ strands entering from the top and } n \text{ strands exiting from the bottom. The top and bottom strands are connected by } m \text{ crossings.} \\
 & = q^{-\frac{m}{3}(n^2+3n)} \sum_{n \geq k_1 \geq k_2 \geq \dots \geq k_m \geq 0} q^{\frac{n-k_m}{2}} q^{\sum_{i=1}^m (k_i^2 + k_i)} \\
 & \quad \times \frac{(q)_n^2}{(q)_{n-k_1} (q)_{k_1-k_2} \dots (q)_{k_{m-1}-k_m} (q)_{k_m}^2} \text{Diagram: A braiding diagram with } n \text{ strands, } m \text{ crossings, and } k_i \text{ strands between crossings.} \\
 & \quad \left(\text{Diagram: A crossing with } q \text{ on the top strand and } q^{-1} \text{ on the bottom strand.} = \text{Diagram: A crossing with } q \text{ on the top strand and } q^{-1} \text{ on the bottom strand.} \right)
 \end{aligned}$$

ツイスト公式の格子の経路を用いた証明方法

e.g.

$$\begin{aligned}
 \text{Diagram 1} &= q^{\frac{3}{4}} \text{Diagram 2} + q^{-\frac{3}{4}} \text{Diagram 3} \\
 &= q^{\frac{3}{4}} \left(q^{\frac{1}{4}} \text{Diagram 4} + q^{-\frac{1}{4}} \text{Diagram 5} \right) + q^{-\frac{3}{4}} \left(q^{\frac{1}{4}} \text{Diagram 6} + q^{-\frac{1}{4}} \text{Diagram 7} \right)
 \end{aligned}$$

skein tree



coefficient of basis web on (1,1)

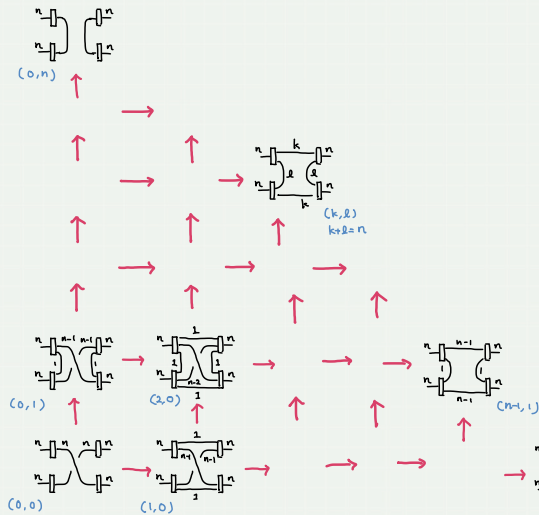
$$= \sum_{\gamma: \text{paths from } (0,0) \text{ to } (1,1)} \prod w$$

w : coeff. on γ

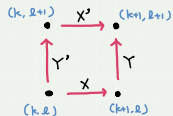
$$= \frac{q^{-\frac{3}{4}} q^{\frac{1}{4}}}{\textcircled{1}} + \frac{q^{\frac{3}{4}} q^{-\frac{1}{4}}}{\textcircled{2}}$$

$\times q$

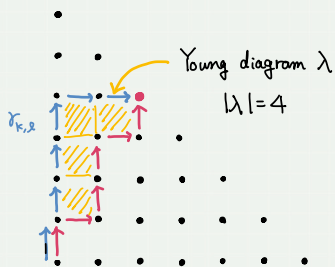
(half twist formula)



Lemma



then $XY = \varrho Y'X'$



$$\text{coeff}_r \left(\begin{array}{c} \nearrow \\ \nearrow \\ \nearrow \\ \nearrow \end{array} \right) = q^{\frac{|\lambda|}{2}} \left(\begin{array}{c} \nearrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array} \right)$$

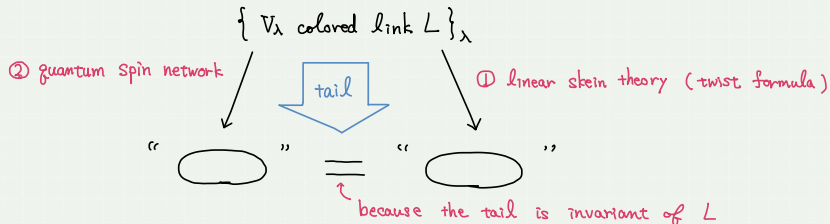
① coefficient of (k, l) ($k+l=n$)

$$= \sum_{\gamma: \text{path from } (0,0) \text{ to } (k,l)} \prod_{w: \text{weight on } \gamma} w$$

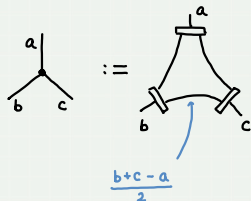
$$= \left(\prod_{w: \text{weight on } \gamma_{k,l}} w \right) \left(\sum_{\substack{\lambda: \text{Young diagram} \\ \# \text{ row} \leq k \\ \# \text{ column} \leq l}} q^{|\lambda|} \right)$$

$$= \left(\prod_{w: \text{weight on } \gamma_{k,l}} w \right) \frac{(q)_n}{(q)_k (q)_{n-k}}$$

結び目の tail と q -級数の恒等式



① quantum spin network ($\mathfrak{g} = \mathfrak{sl}_2$)



$$\begin{array}{c} b \\ \diagdown \\ \bullet \\ \diagup \\ a \end{array} \begin{array}{c} i \\ \diagdown \\ \bullet \\ \diagup \\ d \end{array} \begin{array}{c} c \\ \diagdown \\ \bullet \\ \diagup \\ d \end{array} = \sum_j \left\{ \begin{array}{ccc} a & b & j \\ c & d & i \end{array} \right\}_q \begin{array}{c} b \\ \diagdown \\ \bullet \\ \diagup \\ a \end{array} \begin{array}{c} c \\ \diagdown \\ \bullet \\ \diagup \\ d \end{array}$$

\uparrow
 quantum 6_j -symbol

$$\begin{array}{c} c \\ \diagdown \\ \bullet \\ \diagup \\ a \end{array} \begin{array}{c} b \\ \diagdown \\ \bullet \\ \diagup \\ a \end{array} = \gamma_a^{bc} \begin{array}{c} c \\ \diagdown \\ \bullet \\ \diagup \\ a \end{array} \begin{array}{c} b \\ \diagdown \\ \bullet \\ \diagup \\ a \end{array}$$

(false) theta series に関する Andrews-Gordon 型の恒等式

$$\bullet f(a, b) := \sum_{i=0}^{\infty} a^{\frac{i(i+1)}{2}} b^{\frac{i(i-1)}{2}} + \sum_{i=1}^{\infty} a^{\frac{i(i-1)}{2}} b^{\frac{i(i+1)}{2}} \quad : \text{the theta series}$$

$$\bullet \Psi(a, b) := \sum_{i=0}^{\infty} a^{\frac{i(i+1)}{2}} b^{\frac{i(i-1)}{2}} - \sum_{i=1}^{\infty} a^{\frac{i(i-1)}{2}} b^{\frac{i(i+1)}{2}} \quad : \text{the false theta series}$$

Theorem[Armond-Dasbach, 2011]

$$f(-q^{2m}, -q)/(1-q) = \mathcal{J}_{T(2, 2m+1)}^{al_2}(q) = \frac{(q)_{\infty}}{1-q} \sum_{k_1 \geq k_2 \geq \dots \geq k_{m-1} \geq 0} \frac{q^{\sum_{i=1}^{m-1} k_i^2 + k_i}}{(q)_{k_1 - k_2} (q)_{k_2 - k_3} \dots (q)_{k_{m-2} - k_{m-1}} (q)_{k_{m-1}}}$$

Theorem[Hajij, 2015]

$$\Psi(q^{2m-1}, q)/(1-q) = \mathcal{J}_{T(2, 2m)}^{al_2}(q) = \frac{(q)_{\infty}}{1-q} \sum_{k_1 \geq k_2 \geq \dots \geq k_m \geq 0} \frac{q^{-k_m} q^{\sum_{i=1}^m k_i^2 + k_i}}{(q)_{k_1 - k_2} (q)_{k_2 - k_3} \dots (q)_{k_{m-1} - k_m} (q)_{k_m}^2}$$

Theorem[Y. 2018]

$$\begin{aligned} \sum_{i=0}^{\infty} q^{-2i} q^m (i^2+2i) \frac{(1-q^{i+1})^3 (1+q^{i+1})}{(1-q)^2 (1-q^2)} &= \mathcal{J}_{\mp}^{\text{sl}_3}(2, 2m)(q) \\ &= \frac{(q)_{\infty}}{(1-q)(1-q^2)} \sum_{k_1 \geq k_2 \geq \dots \geq k_m \geq 0} \frac{q^{-2k_m} q^{\sum_{i=1}^m k_i^2 + 2k_i}}{(q)_{k_1-k_2} (q)_{k_2-k_3} \dots (q)_{k_{m-1}-k_m} (q)_{k_m}^2} \end{aligned}$$

Theorem[Y. 2020]

$$\mathcal{J}_{\mp}^{\text{sl}_3}(2, 2m+1)(q) = \frac{f(-q^{2m}, -q)}{(1-q)^2 (1-q^2)} \quad \mathcal{J}_{\mp}^{\text{sl}_3}(2, 2m)(q) = \frac{\Psi(q^{2m-1}, q)}{(1-q)^2 (1-q^2)}$$

$$\rightsquigarrow \mathcal{J}_{\mp}^{\text{sl}_3}(2, m)(q) = \frac{1}{(1-q)(1-q^2)} \mathcal{J}_{\mp}^{\text{sl}_2}(2, m)(q)$$