Title : Skein and cluster algebras of marked surfaces without punctures for $\mathfrak{sl}_3$

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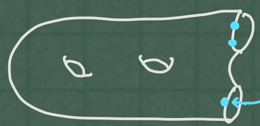
2021-11-16 トポロジー火曜セミナー

- Σ : a marked surface  $\mathbb{M}$: a set of marked points

- $\mathcal{S}_{\mathfrak{g},\Sigma}$: the skein algebra of a ori. surface Σ associated with $\mathfrak{g}$.

  quotient alg. of tangles in $\Sigma \times [0,1]$

  skein rel. of $\mathfrak{g}$

- $\mathcal{A}_{\mathfrak{s}_\mathfrak{g}(\mathfrak{g},\Sigma)}$: the quantum cluster algebra of Σ ass. with $\mathfrak{g}$.

  quiver (ideal triangulations) + mutations

  Laurent phenomenon (Berenstein - Zelevinsky '05)

- $\mathcal{U}_{\mathfrak{s}_\mathfrak{g}(\mathfrak{g},\Sigma)}$: the upper quantum cluster alg. ——

  Laurent polynomials of cluster variables.

<u>Theorem</u> (Muller 2016, $\mathfrak{g} = \mathfrak{sl}_2$)

$$\underbrace{\mathcal{A}_{\mathfrak{S}_q(\mathfrak{sl}_2,\Sigma)} \subset \mathcal{S}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] \subset \mathcal{U}_{\mathfrak{S}_q(\mathfrak{sl}_2,\Sigma)}}_{\mathcal{A} = \mathcal{U} \text{ if } \#\mathbb{M} \geq 2} \subset \operatorname{Frac} \mathcal{S}_{\mathfrak{sl}_2,\Sigma}$$

<u>Main Results</u> (Ishibashi - Y. '21, $\mathfrak{g} = \mathfrak{sl}_3$, ($\mathfrak{sp}_4$))

$\mathfrak{g} = \mathfrak{sl}_3$ $\quad \#\mathbb{M} \geq 2$

($\mathfrak{sp}_4$) ① $\mathcal{S}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \subset \mathcal{A}_{\mathfrak{S}_q(\mathfrak{sl}_3,\Sigma)} \subset \mathcal{U}_{\mathfrak{S}_q(\mathfrak{sl}_3,\Sigma)} \subset \operatorname{Frac} \mathcal{S}_{\mathfrak{sl}_3,\Sigma}$

"sticking trick"

"cutting trick"

② "elevation preserving web" in $\mathcal{S}_{\mathfrak{sl}_3,\Sigma}$ has a positive Laurent expression in $\mathcal{U}_{\mathfrak{S}_q(\mathfrak{sl}_3,\Sigma)}$ via the cutting trick

<u>Remark</u> If covering conjecture

「$\mathcal{S}_{\mathfrak{sl}_3,\Sigma} = \bigcap_{E} \mathcal{S}_{\mathfrak{sl}_3,\Sigma}[(\Delta \setminus E)^{-1}]$」 is true,

we have $\mathcal{S}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] = \mathcal{U}_{\mathfrak{S}_q(\mathfrak{sl}_3,\Sigma)}$

§ Comparison of cluster & skein algebras for $\mathfrak{g} = \mathfrak{sl}_2$

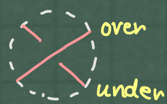
- Σ : a marked surface

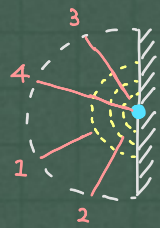
- $\mathrm{Tang}(\Sigma) := \{$ tangle diagrams on Σ $\}$

  //

  generic immersed arcs & loops γ in Σ $(\partial\gamma \in \mathbb{M})$

  with $\left|\begin{array}{l} \text{over / under-passing information at crossing points.} \\ \text{“elevation” at marked points} \end{array}\right.$





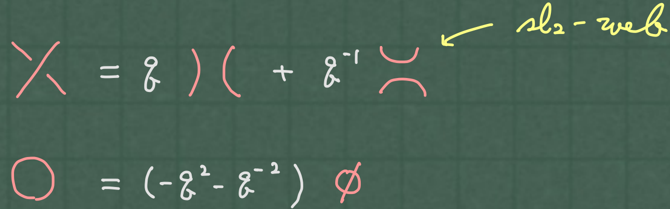
  multiplication of $T_1, T_2 \in \mathrm{Tang}(\Sigma)$ $\quad T_1 \cdot T_2$

  T_1: over (higher) $\quad T_2$: under (lower)

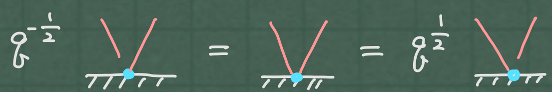
- Skein algera $\mathcal{S}_{\mathfrak{sl}_2, \Sigma} := \mathbb{Z}[q^{\pm 1/2}]\,\mathrm{Tang}(\Sigma)\,/\,$skein relations

  Def. (skein relations for $\mathfrak{sl}_2$)

  – the Kauffman bracket skein relation



  – Muller's boundary skein relation



Thm (Muller '16) $\mathcal{S}_{\mathfrak{sl}_2, \Sigma}$ is an Ore domain

$\rightsquigarrow$ $\mathcal{S}_{\mathfrak{sl}_2, \Sigma}[\partial^{-1}] \subset \mathrm{Frac}\, \mathcal{S}_{\mathfrak{sl}_2, \Sigma}$

(∂ : the set of boundary webs )

⑪ the cluster alg. $\mathcal{A}_{s_q(sl_2,\Sigma)}$ in $\mathrm{Frac}\,\mathcal{S}_{sl_2,\Sigma}$

Δ : an ideal triangulation of Σ

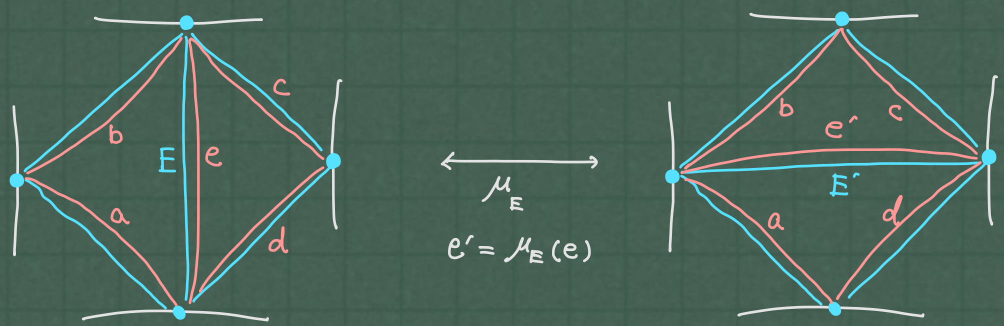
$\longleftrightarrow$ $\mathcal{C}_\Delta = \{ \underline{\text{simple arcs}}$ along $\Delta \} \subset \mathcal{S}_{sl_2,\Sigma}$: a cluster

cluster variables

Δ' is obtained by a flip of Δ at E

$\longleftrightarrow$ $\mathcal{C}_\Delta$ is related to $\mathcal{C}_{\Delta'}$ by mutation

$\quad\quad$ " skein rel.



$$e\,e' = q\,ac + q^{-1}bd$$: Exchange relation

only have to x = simple loop

If $\forall x \in \mathcal{S}_{sl_2,\Sigma}[\partial^{-1}]$ is expressed as a polynomial of simple arcs and inverses of ∂.

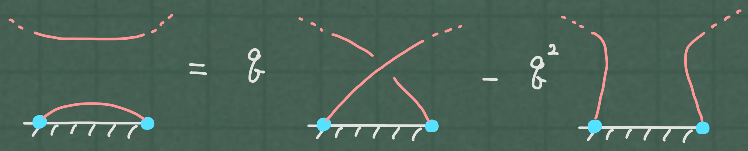
then,

$$\begin{array}{ccc} \mathcal{A}_{s_q(sl_2,\Sigma)} & \subset & \\ \cup & & \mathrm{Frac}\,\mathcal{S}_{sl_2,\Sigma} \\ \mathcal{S}_{sl_2,\Sigma}[\partial^{-1}] & \subset & \end{array}$$

use "sticking trick"

◎ Cutting & sticking tricks

<u>Lem</u> (the sticking trick) $\mathcal{S}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] \longrightarrow \mathcal{A}_{\mathcal{S}_q(\mathfrak{sl}_2,\Sigma)}$



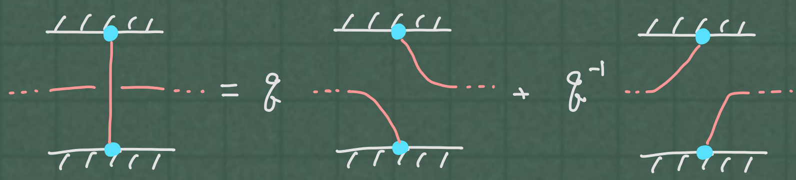
<u>e.g.</u> (an expansion of a simple loop γ)

$$= q^2 \,[\cdot]\, - q^3 \,[\cdot]\, - q^3 \,[\cdot]\, + q^4 \,[\cdot]$$

$\mathcal{S}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}]$

$\therefore \quad x = $ a polynomial in simple arcs $\Big/ \prod_{e_i \in \partial} e_i^{n_i} \in \mathcal{A}_{\mathcal{S}_q(\mathfrak{sl}_2,\Sigma)}$

<u>Lem</u> (the cutting trick) $\mathcal{S}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] \longrightarrow \mathcal{U}_{\mathcal{S}_q(\mathfrak{sl}_2,\Sigma)}$



$\rightsquigarrow \quad x = $ a Laurent polynomial in $\underbrace{\text{arcs along } \Delta}_{\text{the cluster } \mathcal{C}_\Delta}$

if $x = $  or  $\Rightarrow$ coefficients in $\mathbb{Z}_{\geq 0}[q^{\pm 1/2}]$
(positivity)

§ Comparison of cluster & skein algebras for $\mathfrak{g} = \mathfrak{sl}_3$

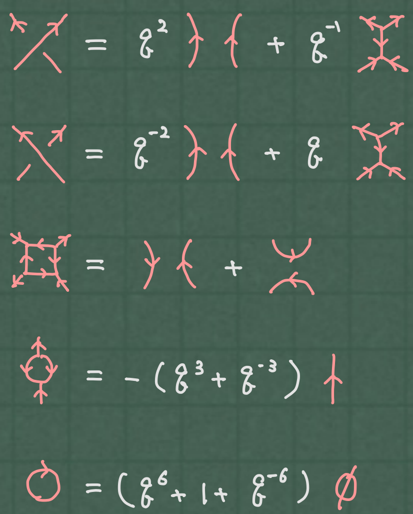
◎ The $\mathfrak{sl}_3$-skein algebra of Σ.

$\mathrm{Tang}(\Sigma) := \left\{ \begin{array}{l} \text{knotted uni-trivalent graph on } \Sigma \\ \text{with} \end{array} \right\}$

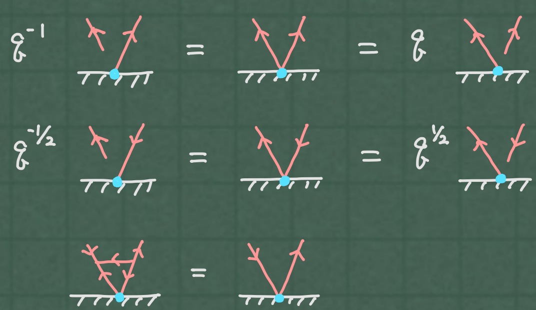
$\mathcal{S}_{\mathfrak{sl}_3, \Sigma} := \mathbb{Z}[q^{\pm 1/2}]\,\mathrm{Tang}(\Sigma) \,/\, \text{skein relations}$

<u>Def.</u> (skein relations for $\mathfrak{sl}_3$)

– Kuperberg's A_2-skein rel.

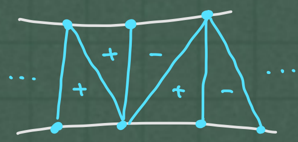


– Frohman-Sikora's skein relation. (at $a=1$)



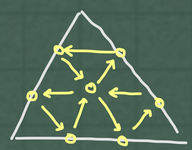
⑩ clusters associated with Δ

$$\mathcal{C}_{\Delta} = \bigcup_{\substack{T:\text{ triangle}\\ \varepsilon\in\{+,-\}}} \mathcal{C}_{T^{\varepsilon}}$$

$\Delta =$ 

$\mathcal{C}_{\Delta^{+}} = \{$  $\}$

$\mathcal{C}_{\Delta^{-}} = \{$  $\}$



quiver

$\cap$

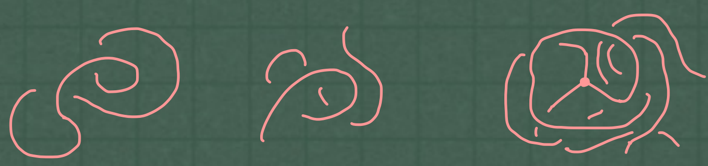
$\mathcal{A}_{S_q(sl_3,\Sigma)}$

Remark $\exists$ clusters do not come from Δ

⑪ Expansion of $\mathscr{S}_{sl_3,\Sigma}[\partial^{-1}]$ into $\mathcal{A}_{S_q(sl_3,\Sigma)}$

Thm (Frohman-Sikora '20)

$\mathscr{S}_{sl_3,\Sigma}$ is generated by descending knots, arcs, and triads.



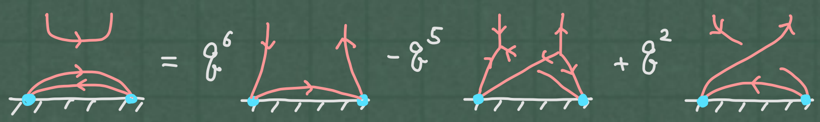
Thm (Ishibashi-Y.)

$\mathscr{S}_{sl_3,\Sigma}[\partial^{-1}]$ is generated by simple arcs & simple triads

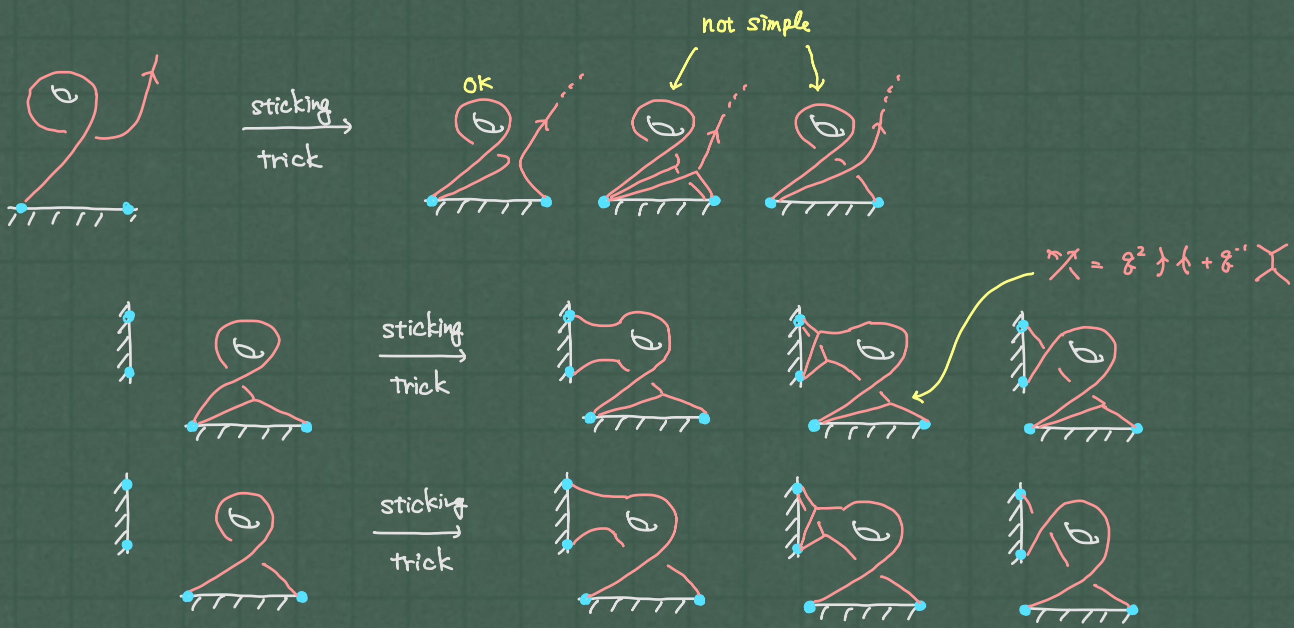
cluster variables

i.e. $\mathscr{S}_{sl_3,\Sigma}[\partial^{-1}] \subset \mathcal{A}_{S_q(sl_3,\Sigma)}$

Lem (sticking trick)

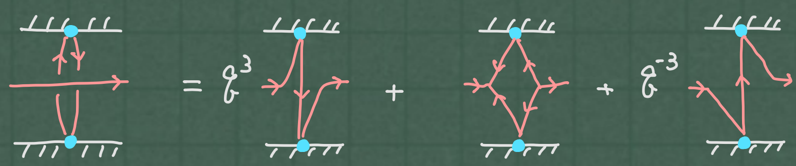


(Sketch of proof of Thm)



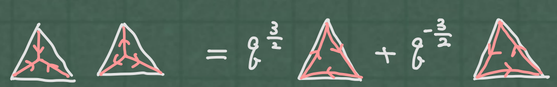
◎ Positivity

Lem (Cutting trick)



$\rightsquigarrow \forall x \in \mathscr{S}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}]$

$$\Big(\prod_{e_i \in \Delta} e_i^{n_i}\Big)\, x = \text{a polynomial in } \bigcup_{\Delta} (\mathcal{C}_{\Delta} \cup \mathcal{C}_{\Delta})$$



$\in \mathcal{C}_{\Delta} \cap \mathcal{C}_{\Delta}$

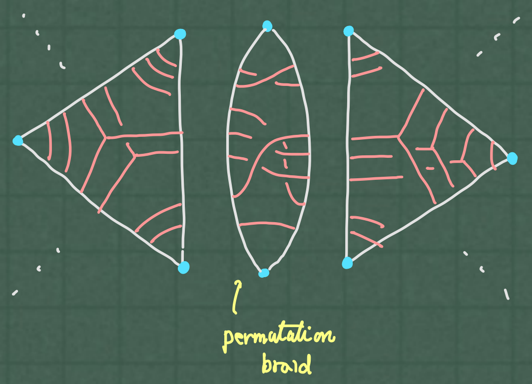
$\therefore\ x = $ a Laurent polynomial in $\mathcal{C}_{\Delta} \in \mathcal{U}_{\mathscr{S}_q(\mathfrak{sl}_3,\Sigma)}$

Thm (Ishibashi - Y.)

For an ideal triangulation Δ,

"elevation preserving" $\mathfrak{sl}_3$-webs w.r.t. Δ

have a positive Laurent expression in $\mathcal{C}_\Delta$

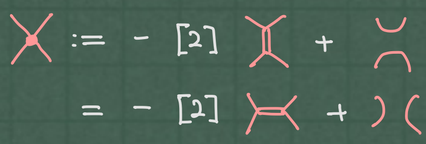


e.g.

are elevation preserving

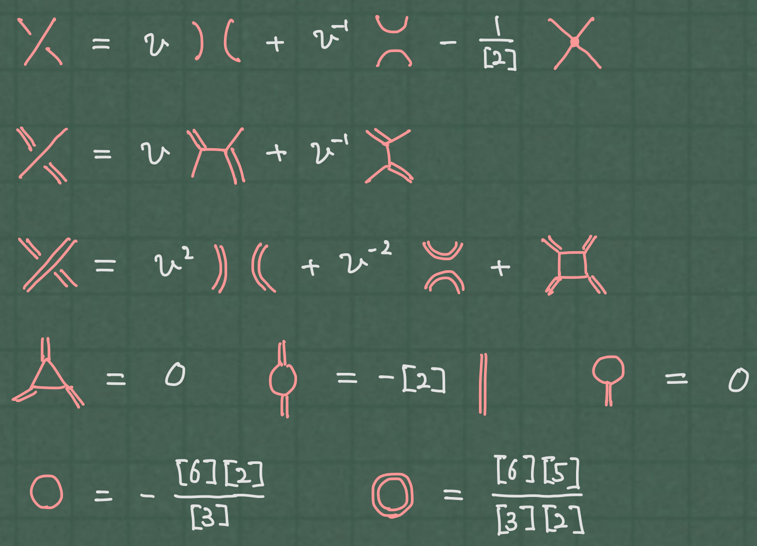
§ $\mathfrak{g} = \mathfrak{sp}_4$

- $\mathfrak{sp}_4$-web 



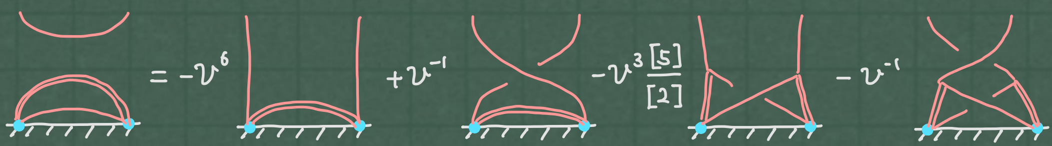
- $\mathfrak{sp}_4$-skein rel.

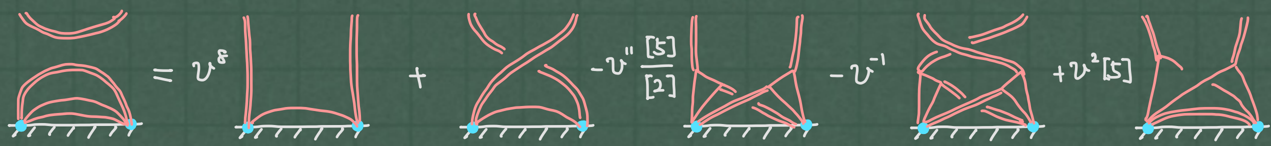
  - Kuperberg's skein rel.



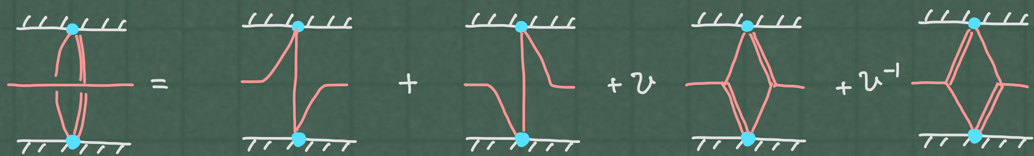
  - boundary skein rel. [Ishibashi - Y. (in prep)]

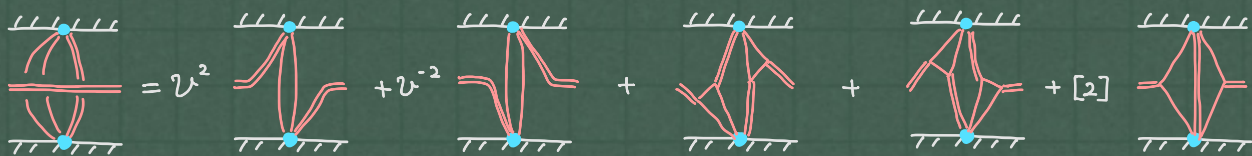
Lem (sticking trick) $\mathscr{S}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}] \longrightarrow \mathcal{A}_{\mathfrak{S}_{\mathfrak{g}}(\mathfrak{sp}_4,\Sigma)}$





Lem (cutting trick) $\mathscr{S}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}] \xrightarrow{\text{pos. Laurent}} \mathcal{U}_{\mathfrak{S}_{\mathfrak{g}}(\mathfrak{sp}_4,\Sigma)}$





- the sticking / cutting tricks work in a similar way ...

[Ishibashi - Y. (in prep.)]