第69回トポロジーシンポジウム
date : 2022-08-17

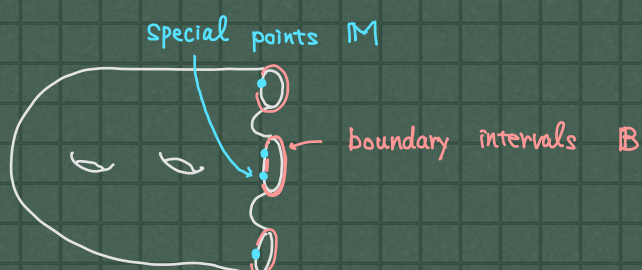
曲面のスケイン代数と量子クラスター代数

湯淺 亘 OCAMI・RIMS

◎ 石橋 典 (東北大) との共著 arXiv:2101.00643 ($\mathfrak{sl}_3$)
arXiv:2207.01540 ($\mathfrak{sp}_4$)
+ in preperation (state-clasp)
に基づく

§ Introduction

unpunctured marked surface $\Sigma =$



+ G

clasped skein algebra $\mathscr{S}^{\mathfrak{z}}_{\Sigma,\mathfrak{g}}$ ⊎ $\mathfrak{g}$-web ↑ skein relations

<u>Conjecture</u> $\mathscr{S}^{\mathfrak{z}}_{\Sigma,\mathfrak{g}}[\partial^{-1}] = \mathscr{A}^{\mathfrak{z}}_{\mathfrak{g},\Sigma}$ ⟷

$\mathfrak{g} = \mathfrak{sl}_2$ Muller (2016)

(quantum) cluster algebra $\mathscr{A}^{\mathfrak{z}}_{\mathfrak{g},\Sigma}$ ⊎ cluster variables ↑ exchange relations

When $\mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sp}_4$, $\#\mathbb{M} \geq 2$

<u>Theorem</u> (Ishibashi-Y.) $\mathscr{S}^{\mathfrak{z}}_{\Sigma,\mathfrak{g}}[\partial^{-1}] \subseteq \mathscr{A}^{\mathfrak{z}}_{\mathfrak{g},\Sigma} \subsetneq \mathrm{Frac}\, \mathscr{S}^{\mathfrak{z}}_{\Sigma,\mathfrak{g}}$

<u>Corollary</u> (Ishibashi-Y. + Ishibashi-Oya-Shen)

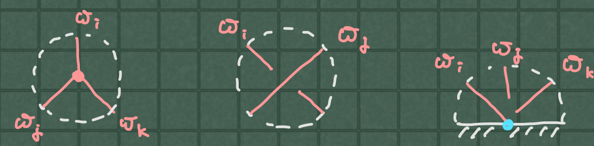
an open subspace of decorated twisted G-local systems on Σ

$$\mathscr{S}^{1}_{\Sigma,\mathfrak{g}}[\partial^{-1}] = \mathscr{A}^{1}_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$$

§ clasped skein algebra $\mathscr{S}^{q}_{\mathfrak{g},\Sigma}$

- $\mathfrak{g}$-webs $\underset{\text{def}}{\Longleftrightarrow}$ tangled uni-trivalent graphs on Σ

with

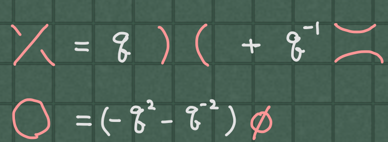


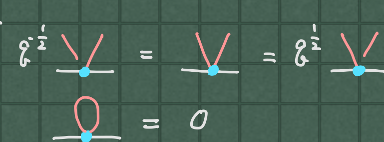
(ϖ_i : fundamental weights)

- skein relations
  - internal : diagrammatic relations in $\mathrm{FundRep}_{\mathfrak{g}}$
    - $\mathfrak{sl}_2$: Kauffman bracket skein relation
    - rank 2 : Kuperberg etc.
  - clasped : diagrammatic relations at Jones-Wenzl projectors
    - $\mathfrak{sl}_2$: Muller
    - $\mathfrak{sl}_3$: Frohman - Sikora
    - $\mathfrak{sp}_4$: Ishibashi - Y.

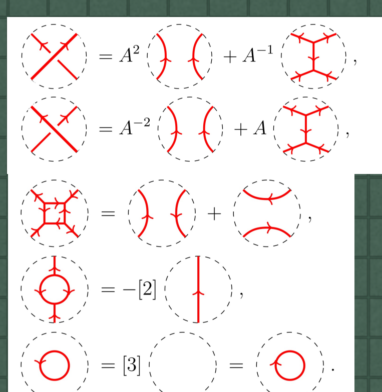
—

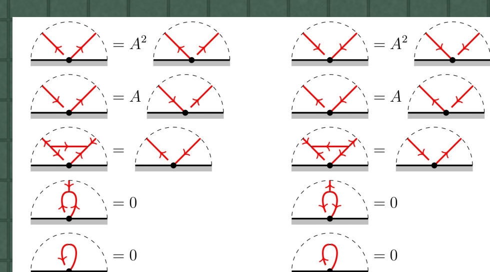
e.g. $\mathfrak{sl}_2$



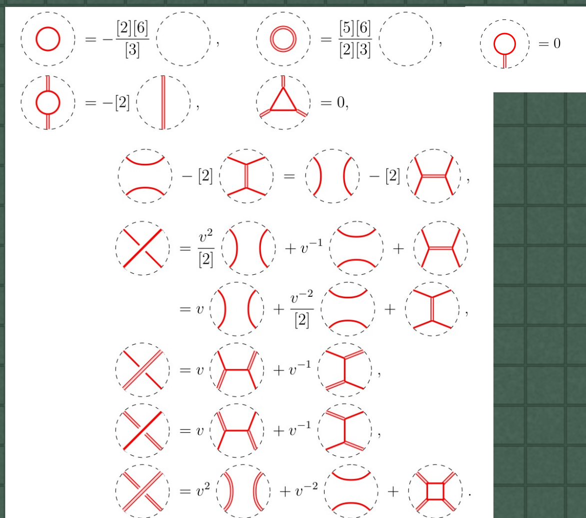


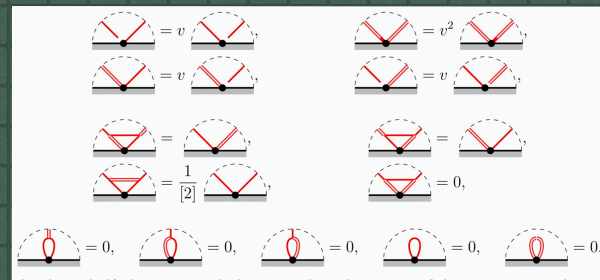
e.g. $\mathfrak{sl}_3$





e.g. sp_4

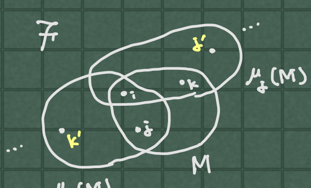




§ quantum cluster algebra $\mathscr{A}^{q}_{\mathfrak{g},\Sigma}$

$\mathcal{F}$: a skew-field, $I = I_{uf} \sqcup I_f$: index set $\quad D = \mathrm{diag}(d_i \in \mathbb{Z}_{>0} \mid i \in I)$

● $S = (B, \Pi, \mathring{\Lambda}, M)$: quantum seed



• $B = (b_{ij})_{i,j\in I} = \begin{array}{cc} \begin{array}{cc} I_{uf} & I_f \end{array} & \\ \left(\begin{array}{c|c} \mathbb{Z} & \mathbb{Z} \\ \hline \mathbb{Z} & \frac{1}{2}\mathbb{Z} \end{array}\right) & \begin{array}{c} I_{uf} \\ I_f \end{array} \end{array}$ s.t. DB : skew-symmetric

exchange matrix

• $\Pi = (\pi_{ij} \in \mathbb{Z})_{i,j\in I}$: skew-symmetric

compatibility matrix

$$A_k A'_k = q^{\bullet} \prod_{j\in I} A_j^{[b_{jk}]_+} + q^{\bullet} \prod_{j\in I} A_j^{[-b_{jk}]_+}$$

quantum exchange relation

• $\mathring{\Lambda} = \bigoplus_{i\in I} \mathbb{Z} f_i$ with a skew-symmetric form $\Pi(f_i, f_j) = \pi_{ij}$

• $M : \mathring{\Lambda} \longrightarrow \mathcal{F} \setminus \{0\}$ s.t. $M(\alpha)M(\beta) = q^{\frac{\Pi(\alpha,\beta)}{2}} M(\alpha+\beta)$

toric frame $\qquad \mathrm{Frac}\, M(\mathring{\Lambda}) = \mathcal{F}$

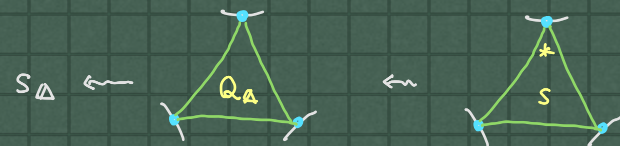
※ $M(f_i) =: A_i$: cluster variable $\begin{cases} \text{unfrozen} & \text{if } i \in I_{uf} \\ \text{frozen} & \text{if } i \in I_f \end{cases}$

invertible

$\{A_i\}_{i\in I}$: cluster

● quantum seed mutation at $k \in I_{uf}$ $\quad (B, \Pi, \mathring{\Lambda}, M) \overset{\mu_k}{\longleftrightarrow} (B', \Pi', \mathring{\Lambda}', M')$

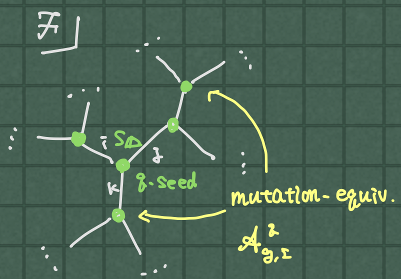
$\mathcal{A}^{q}_{\mathfrak{g},\Sigma} \underset{\text{def}}{:\Longleftrightarrow}$ $\mathbb{Z}_q = \mathbb{Z}[q^{\pm 1/2}]$ subalgebra of $\mathcal{F}$ generated by all quantum seeds mutation-equivalent to quantum seed associated with <u>decorated triangulation $\boldsymbol{\Delta}$</u> of Σ.



(weighted) quiver

decorated triangulation $\boldsymbol{\Delta}$

s : reduced word of the longest element of $W(G)$



$I_{\boldsymbol{\Delta}}$ = vertices of $Q_{\boldsymbol{\Delta}}$

frozen var. = vertices on $\mathbb{B}$

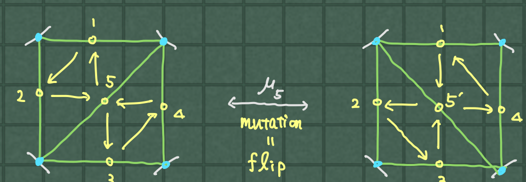
※ $\forall\, \boldsymbol{\Delta}, \boldsymbol{\Delta}'$: $S_{\boldsymbol{\Delta}}$ is mutation-equivalent to $S_{\boldsymbol{\Delta}'}$

$\left.\begin{array}{l} \text{① Construct seeds } S_{\boldsymbol{\Delta}} \text{ \& adjacent ones in } \mathcal{F} = \operatorname{Frac} \mathscr{S}^{q}_{\mathfrak{g},\Sigma} \\ \quad + \text{ quantum Laurent phenomenon} \end{array}\right\} \rightsquigarrow \mathcal{A}^{q}_{\mathfrak{g},\Sigma} \subset \operatorname{Frac} \mathscr{S}^{q}_{\mathfrak{g},\Sigma}$

---

e.g. $\mathfrak{sl}_2$ $\quad$ $\boldsymbol{\Delta} = \Delta$: ideal triangulation

all seeds are associated with Δ's



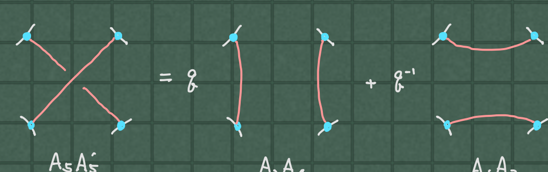
$\mathcal{A}^{q}_{\mathfrak{sl}_2,\triangle}$ = quantum torus

$\mathcal{A}^{q}_{\mathfrak{sl}_2,\square} = \langle A_1, \dots, A_4, A_5 \rangle \cup \langle A_1', \dots A_4', A_5' \rangle$

$A_5 A_5' = A_2 A_4 + A_1 A_3$

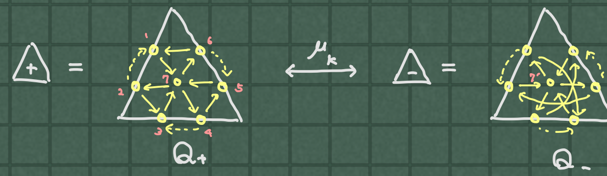
$A_i = A_i' \quad (i \neq 5)$

② In $\mathcal{F} = \operatorname{Frac} \mathscr{S}^{q}_{\mathfrak{sl}_2,\Sigma}$ $\quad A_i$ = simple arc along



$\left\{\begin{array}{l} \mathscr{S}^{q}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] \text{ is generated by simple arcs } \rightsquigarrow \mathscr{S}^{q}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] \subseteq \mathcal{A}^{q}_{\mathfrak{sl}_2,\Sigma} \\ \forall \text{ cluster variables correspond to simple arcs } \rightsquigarrow \mathcal{A}^{q}_{\mathfrak{sl}_2,\Sigma} \subseteq \mathscr{S}^{q}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] \end{array}\right.$

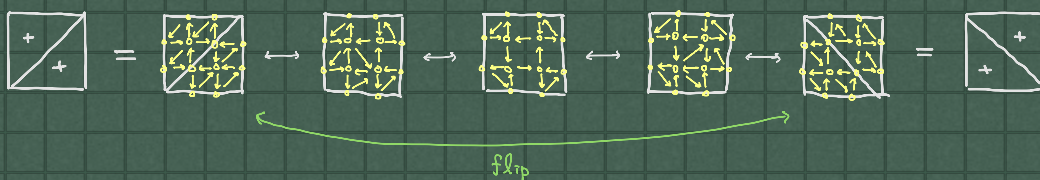
e.g. $\mathfrak{sl}_3$ decorated triangles



$$\mathcal{A}^q_{\mathfrak{sl}_3,\Delta} = \langle A_1,\dots,A_6,A_7\rangle \cup \langle A_1',\dots,A_6',A_7'\rangle$$

$$\begin{cases} A_i' = A_i & (i \neq 7) \\ A_7A_7' = q^\circ A_1A_3A_5 + q^\circ A_2A_4A_6 \end{cases}$$

☆ mutation $\neq$ flip



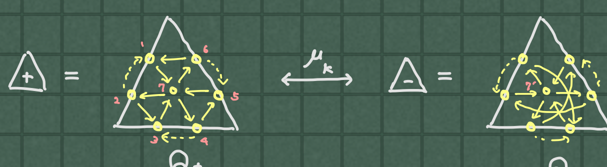
☆ $\mathcal{A}^q_{\mathfrak{sl}_3,\square}$ has 50 clusters (type D_4)

in general, $\mathcal{A}^q_{\mathfrak{sl}_3,\Sigma}$ is infinite mutation type

---

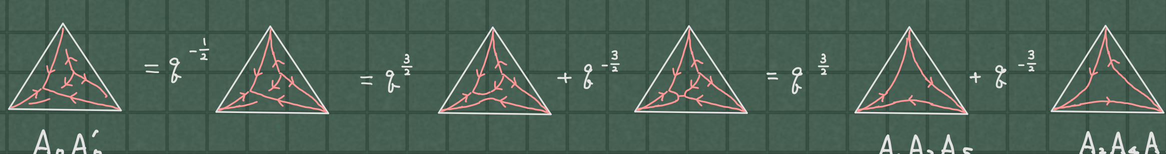
- In $\mathcal{F} = \mathrm{Frac}\,\mathcal{S}^q_{\mathfrak{sl}_3,\Sigma}$

S_Δ :

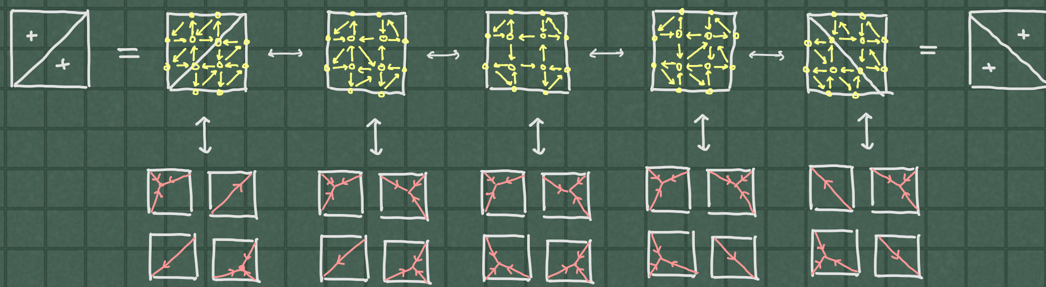


$A_7A_7' = q^\circ A_1A_3A_5 + q^\circ A_2A_4A_6$ from exchange relation.

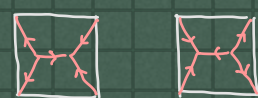
By skein relation :



⑩ a flip sequens in $\mathrm{Frac}\,\mathscr{S}^{q}_{\mathfrak{sl}_3,\Sigma}$



⑪ other cluster variables in $\square$



<u>Theorem</u> (Ishibashi–Y. '21)

matrix elements of simple Wilson lines

$\mathscr{S}^{q}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}]$ is generated by the above cluster variables in ideal quadrilaterals

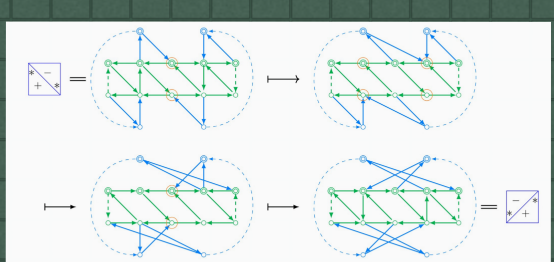
$\rightsquigarrow \mathscr{S}^{q}_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \subseteq \mathcal{A}^{q}_{\mathfrak{sl}_3,\Sigma}$

<u>proof</u> By the sticking trick.

<u>e.g.</u> $\mathfrak{sp}_4$ decorated triangulations

weighted quiver

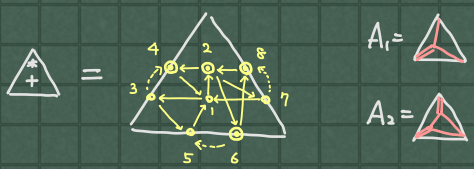
☆ a flip is realized by 8 mutations

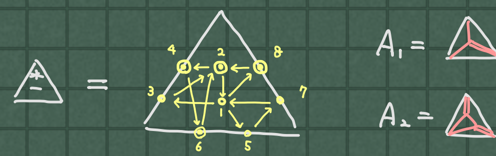


$\mu_1 : A_1 A_1' = q^{\bullet} A_2 A_3 + q^{\bullet} A_4 A_5 A_7$

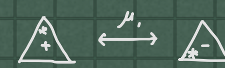
☆ $\mathcal{A}^{q}_{\mathfrak{sp}_4,\square}$ is infinite mutation type

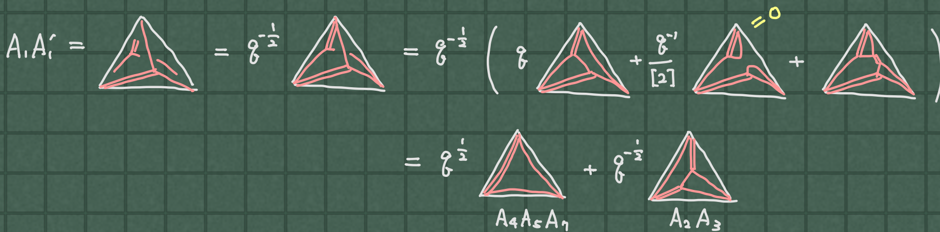
In $\mathcal{F} = \operatorname{Frac} \mathscr{S}^{q}_{\mathfrak{sp}_4,\Sigma}$





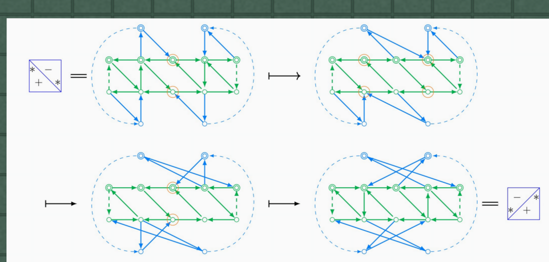
$\mu_1 : A_1 A_1' = q^{\circ} A_2 A_3 + q^{\bullet} A_4 A_5 A_7$ from exchange relation

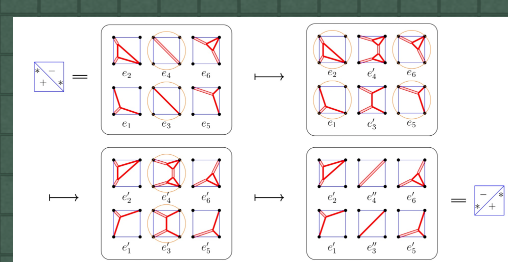




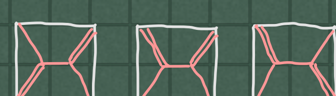
—

⊘ a flip sequence in $\operatorname{Frac} \mathscr{S}^{q}_{\mathfrak{sp}_4,\Sigma}$





⊘ other cluster variables in $\operatorname{Frac} \mathscr{S}^{q}_{\mathfrak{sp}_4,\square}$:



Theorem (Ishibashi – Y. '22)

$\mathscr{S}^{q}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}]$ is generated by the above cluster variables in ideal quadrilaterals

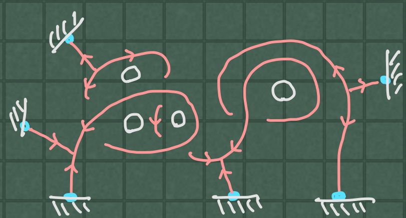
proof By the sticking trick. $\rightsquigarrow \mathscr{S}^{q}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}] \subseteq \mathscr{A}^{q}_{\mathfrak{sp}_4,\Sigma}$

§ generators of $\mathcal{S}^{\mathfrak{g}}_{\mathfrak{g},\Sigma}$

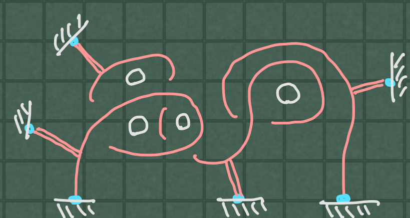
<u>Theorem</u> (descending generators for $\mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sp}_4$)

$\mathcal{S}^{\mathfrak{g}}_{\mathfrak{g},\Sigma}$ is generated by descending curves with / without legs.

<u>$\mathfrak{sl}_3$</u>



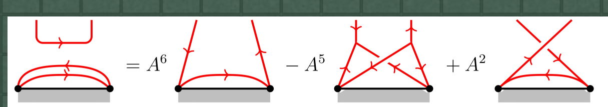
<u>$\mathfrak{sp}_4$</u>



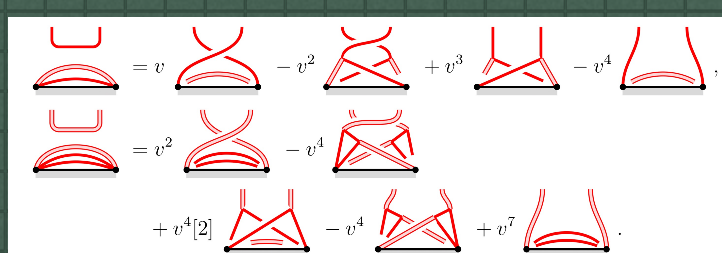
descending generators $\xrightarrow[\text{sticking trick}]{\text{in } \mathcal{S}^{\mathfrak{g}}_{\mathfrak{g},\Sigma}[\partial^{-1}]}$ simple Wilson lines

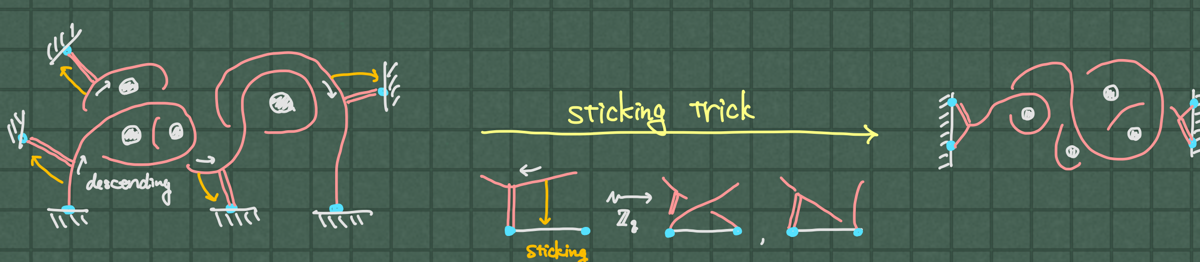
<u>Lemma</u> (the sticking trick) [Ishibashi - Y. '21 '22]

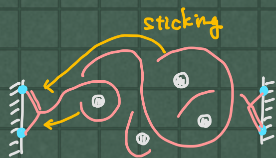
- $\mathfrak{sl}_3$



- $\mathfrak{sp}_4$

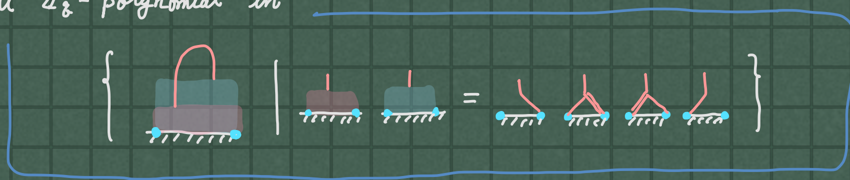


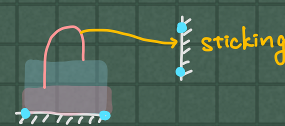




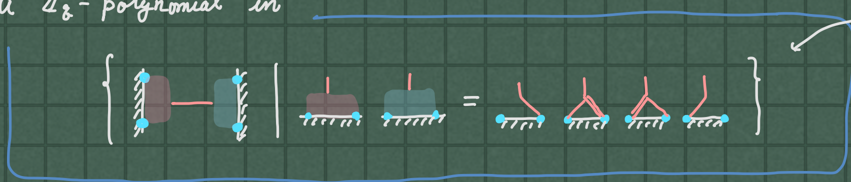
sticking trick

a $\mathbb{Z}_\delta$-polynomial in



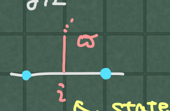


a $\mathbb{Z}_\delta$-polynomial in



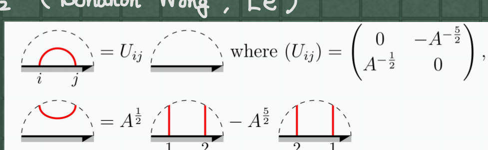
§ state-clasp correspondence

◍ the stated skein algebra $\mathcal{S}^{\delta}_{\mathfrak{g},\Sigma}(\mathbb{B})$

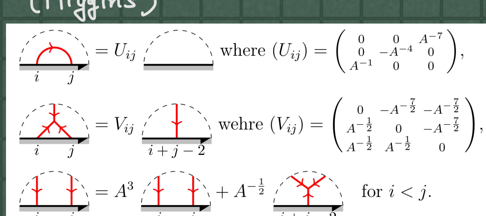
$\underset{\text{def}}{\Longleftrightarrow}$ $\mathfrak{g}$-webs with  $(i \in \Lambda_{\varpi})$

+ internal & stated skein relations

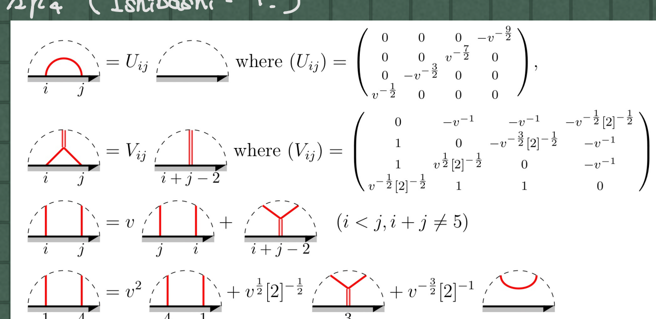
- $\mathfrak{sl}_2$ (Bonahon-Wong, Lê)



- $\mathfrak{sl}_3$ (Higgins)

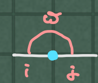


- $\mathfrak{sp}_4$ (Ishibashi - Y.)



◎ the reduced stated skein algebra $\mathscr{S}^{q}_{\mathfrak{g},\Sigma}(\mathbb{B})_{\mathrm{rd}}$

$$\mathscr{S}^{q}_{\mathfrak{g},\Sigma}(\mathbb{B})_{\mathrm{rd}} := \mathscr{S}^{q}_{\mathfrak{g},\Sigma}(\mathbb{B}) / I_{\mathrm{bad}}$$

$I_{\mathrm{bad}} := \mathrm{span}_{\mathbb{Z}_q}\{$  $\mid i < j$ for $i, j \in \Lambda_{\varpi} \}$

☆ What is the stated arc 

→ the (i,j)-entry of a monodromy along  of the moduli space $\mathcal{A}_{G,\Sigma}$ of decorated twisted G-local systems

e.g. $\mathcal{O}(G) \cong \mathscr{S}^{q}_{\mathfrak{g},\circ}(\mathbb{B})$ $(\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3 . (\mathfrak{sp}_4))$

Theorem (the state-clasp correspondence)

$$\underset{\text{clasped}}{\mathscr{S}^{q}_{\mathfrak{g},\Sigma}[\partial^{-1}]} \cong \underset{\text{stated}}{\mathscr{S}^{q}_{\mathfrak{g},\Sigma}(\mathbb{B})_{\mathrm{rd}}}$$

$(q=1)$

q: generic

$$\begin{tikzcd}
\mathscr{S}_{\mathfrak{g},\Sigma}[\partial^{-1}] \arrow[r, "\cong"', "\text{sticking}"'] \arrow[d, shift right, "\mathscr{S}_{\mathbb{M}\to\mathbb{B}}"'] & \mathcal{A}_{\mathfrak{g},\Sigma} \arrow[r, "\cong"] & \mathcal{U}_{\mathfrak{g},\Sigma} \arrow[d, "\cong"', "\text{Ishibashi-Oya-Shen}"] \\
\mathscr{S}^{q}_{\mathfrak{g},\Sigma}(\mathbb{B})_{\mathrm{rd}} \arrow[u, shift right, "\mathscr{S}_{\mathbb{B}\to\mathbb{M}}\ \text{state-clasp}"', "\cong"] & & \mathcal{O}(\mathcal{A}_{\mathfrak{g},\Sigma}) \arrow[ll, "?", "(i,j)\text{-entry of Wilson lines}"']
\end{tikzcd}$$

⊘ $\mathfrak{sl}_2$

$S_{M\to B}$:

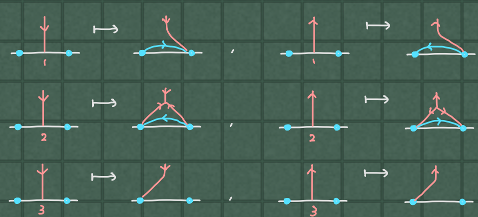
maximum of states

$S_{B\to M}$:

⊘ $\mathfrak{sl}_3$

$S_{M\to B}$:

$S_{B\to M}$:



⊘ $\mathfrak{sp}_4$

$S_{M\to B}$:

$S_{B\to M}$: